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Deformations of closed strings and topological open membranes

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Abstract: We study deformations of topological closed strings. A well-known example is the perturbation of a topological closed string by itself, where the associative OPE product is deformed, and which is governed by the WDVV equations. Our main interest will be closed strings that arise as the boundary theory for topological open membranes, where the boundary string is deformed by the bulk membrane operators. The main example is the topological open membrane theory with a nonzero 3-form field in the bulk. In this case the Lie bracket of the current algebra is deformed, leading in general to a correction of the Jacobi identity. We identify these deformations in terms of deformation theory. To this end we describe the deformation of the algebraic structure of the closed string, given by the BRST operator, the associative product and the Lie bracket. Quite remarkably, we find that there are three classes of deformations for the closed string, two of which are exemplified by the WDVV theory and the topological open membrane. The third class remains largely mysterious, as we have no explicit example.

Keywords: Topological Field Theories, Chern-Simons Theories, Conformal Field Models in String Theory, Differential and Algebraic Geometry.

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Contents

1. Introduction

In recent years there has been much interest in string theory towards noncommutative geometry, and noncommutative gauge theories. It was found at first that noncommutative gauge theories gave a natural description of M-theory in the presence of a NS B-field [1], or more generally for D-branes in the presence of a B-field [2]–[5]. This noncommutative gauge theory on the D-branes can be understood as a description of the open string (field) theory in a decoupling limit. Rather than being a special situation in open string theory, noncommutativity seems to be quite generic, and is closely connected to the extended nature of strings.

The noncommutative star product can be understood in terms of deformation quantisation: a deformation of function algebras starting from a Poisson bracket [6, 7]. This mathematical problem was solved recently by Tamarkin [8], and by Kontsevich and Soibelman [9]. The explicit solution found by Kontsevich can be understood quite elegantly in terms of the perturbation theory of a particular simple topological string [10]. In our paper [11] we considered formal generalisations of these deformations for general topological open strings. The deformation quantisation problem is a special case of the more general problem of deformations of associative algebras. The results showed that the problems of deforming an associative algebra and string theory are intimately connected. This parallels the Deligne "conjecture" in mathematics (see e.g. [7]), which states that the deformation theory of a "1-algebra" is a "2-algebra". In general d-algebras are intimately connected to d-dimensional (topological) field theories: they are defined in terms of (tree level) products for local operators in d dimensions. A 1-algebra is simply an associative algebra. Indeed, we know that point particles are described by quantum mechanics, and operators in quantum mechanics form an associative algebra. On the other hand, string theories (2-dimensional quantum field theories) in general have the structure of a Gerstenhaber algebra — an algebra consisting of a product and a Lie bracket [13]–[16] — which is the same thing as a 2-algebra [7]. Hence the Deligne conjecture can be interpreted as stating that the deformation of a point particle theory is described by a string theory [10, 11]. Indeed, in the case of noncommutative geometry, the boundary theory of the open string, which is a gauge theory, is deformed to a noncommutative gauge theory in the sense of Connes by turning on a B-field, which is a closed string operator coupling to the bulk of the string.

The Deligne "conjecture", which is now proven, can be generalised to higher dimensions [7, 17]: the deformation theory of a d-algebra is conjectured to have the structure of a $(d+1)$-algebra. A natural question from this point of view is therefore whether the deformation theory of 2-dimensional (topological) field theories, or more generally closed string (field) theories, can be described by open membranes.

Parallel to this is the question what the effect is of the 3-form field on the closed string theory. Indeed the natural generalisation of the 2-form coupling to the bulk of the string is the 3-form field in the case of the open membrane. This 3-form field can be interpreted either as the field strength of a 2-form gauge field — which couples to the boundary string as a gauge field — or as the C-field in M-theory, or as the 3-form RR field in type IIA string theory. Attempts to describe the effect in terms of constrained canonical quantisation have been undertaken recently [18, 19, 20]. In these papers a noncommutative deformation of loop space was suggested. A natural situation where the effect of a 3-form occurs is the M-theory membrane ending on a M5-brane. This situation is particularly relevant as it may provide more insight about the still mysterious M5-brane. The place to study the effects are various decoupling limits of the M5-brane theories, in particular the $(2,0)$ little string theory [21, 22], and the recently proposed OM theory point [23, 24]. In these situations the decoupled theories one studies can be interpreted as closed string theories. Moreover, they can be seen as the boundary of the supermembrane. The C-field is a bulk membrane deformation. The effect of this C-field can therefore be interpreted as a deformation of a closed string by an open membrane. Related to this by double dimensional reduction is the type IIA situation of a D2-brane ending on a D4-brane, in a certain decoupling limit [23, 24].

A deformation theory of closed strings, especially in the context of topological string theory, was already studied about a decade ago [25, 26]. However, this deformation theory concerned the deformation of closed strings by the closed string operators themselves.

As deformations of closed strings can come either from closed strings or from membranes, the question arises which of the two describes the proper deformation theory of closed strings. In this paper we study the general deformation complex of closed string theories. We show the connection of the string theory correlation functions and their deformations to the abstract deformation complex. We find that in general deformations of closed strings cannot be described by a single deformation complex. The paper is organised as follows.

In section 2, we discuss two-dimensional topological field theories, whose correlation functions have the structure of a Gerstenhaber algebra. Not restricting to physical operators leads to algebras up to homotopy, defined by higher correlation functions. They contain generalisations of both the associative and the Lie structure, and are combined into a G_∞ algebra. We make a concrete proposal for the A_∞ part of the algebra.

In section 3 we review deformations of the closed string algebra by inserted closed string operators. The associativity of the deformed product is guaranteed by the WDVV equations. We will argue that this goes through for the full G_∞ algebra; it turns out that only the A_∞ structure is deformed. The multilinear maps deforming the products are seen to form a structure of Gerstenhaber algebra themselves.

We show that this is the same algebra as the underlying algebra of the deforming operators. The associativity in first order of the deformed product corresponds to the BRST-closedness of the deforming operator.

In section 4 we describe the general structure of deformations of closed strings. This is governed by the Hochschild complex, which contains all possible deformations of algebraic operations. The Hochschild complex is an algebra by itself, part of whose structure is induced by the algebraic structure that is deformed. In the case of the open string, this structure is determined by the (undeformed) open string theory. For the closed string however, we find that the full structure induced by the undeformed closed string cannot be used to define a consistent deformation theory. One can only consistent deform a substructure. This leads a priori to three different classes of deformation theories, reflected in three different structures of complexes; which one is valid depends of course on the specific model under consideration.

In section 5, we specify the classes of deformation complexes. The deformation of closed strings by themselves studied in Section 3 turns out to have structure of one of these three. The second class, related to deformations of the L_∞ structure, is described by a 3-dimensional theory. This leads us to suggest that it can be understood in terms of topological open membrane theories, where the boundary string is deformed by bulk membrane operators. For the third deformation complex, which should be described by a 2-dimensional theory, we have no explicit realisation.

In section 6 we discuss topological open membranes in a general setting. We try to describe the deformation theory of the boundary string theory by the membrane bulk operators. Though we are not able to prove all Ward identities in detail, due to our lack of understanding 3-dimensional conformal field theories, we argue that indeed the L_∞ structure is deformed, and that the deformation theory has the structure of the second class of deformation complexes.

In section 7 we describe an explicit example for the topological open membrane (TOM), which was defined in [27]: an open membrane with only a WZ term, defined by a closed 3-form field. The undeformed boundary string theory is the closed string version of the Cattaneo-Felder model [10]. The coupling of the bulk membrane to the C-field indeed deforms the closed string Lie bracket. We find that it induces a trilinear operation, which gives a correction to the Jacobi identity of the bracket.

In section 8 we mention some possible extensions and relations to physical models, such as OM theory, self-dual little strings, and M5-branes. On the basis of the structure that we found in the open membrane, we speculate about consistent generalisations of interacting 2-form gauge theories, such as "non-abelian" 2-forms.

2. Topological closed strings

Topological field theories are supplied with a BRST operator Q, an anticommuting scalar, squaring to zero. For the theory to be independent of the metric, the energy-

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momentum tensor T should be BRST-exact. As T generates translations, this implies the existence of an operator G such that

$$\{Q, G\} = d\,. \tag{2.1}$$

For the bosonic string for example, this operator is given by the mode b_1 of the antighost. The operator G is fermionic too and should be a 1-form on the worldvolume. Furthermore, there is a conserved U(1) symmetry, whose conserved charge is called ghost number, such that the BRST operator Q has ghost number 1, and the energy-momentum tensor, along with all physical operators, has ghost number zero. This implies that G has ghost number -1.

Starting from any operator $\alpha \equiv \alpha^{(0)}$ that is a scalar on the worldsheet, one can repeatedly use the operator G to define other operators, denoted $\alpha^{(p)}$, by the relation $\alpha^{(p)} = \{G, \alpha^{(p-1)}\}$. They are called called descendants. As G is a 1-form, the descendant $\alpha^{(p)}$ is a p-form on the worldsheet. Due to the anticommutation relations (2.1), they satisfy the descent equations, $Q\alpha^{(p+1)} = d\alpha^{(p)}$. Using anticommuting coordinates θ^μ on the worldsheet, one can combine the operator α and its descendants into a "superfield", $\boldsymbol{\alpha} = \alpha + \theta\alpha^{(1)} + \frac{1}{2}\theta^2\alpha^{(2)}$, where contractions are suppressed in the notation. The condition for physical or BRST-closed operators α, $Q\alpha = 0$, is now equivalent to closedness of the superfield with respect to the full derivation $Q + D$, where the superderivative operator $D = \theta^\mu\partial_\mu$ is introduced. We will assume that the scalar operator is BRST-closed, unless stated otherwise.

For any operator and its descendants we can build corresponding observables by integrating them. The basic local observable is the evaluation of the operator in a point x, $\alpha(x)$. The descendants give rise to nonlocal observables $\int_{C_p}\alpha^{(p)}$, where in general C_p is a p-cycle in the worldvolume. Note that the second descendant can be used to deform the action, $\delta S = \int_\Sigma \alpha^{(2)}$. The descent equations guarantee that these observables are BRST-closed and only depend on the homology class of the cycle C_p. For example,

$$\alpha(x') - \alpha(x) = \int_x^{x'} d\alpha = \int_x^{x'} \{Q, G\}\alpha = Q\int_x^{x'} \alpha^{(1)}\,, \tag{2.2}$$

which decouples as it is BRST-exact.

Next we discuss the correlation functions in the topological string theory. They can be identified with an algebraic structure on the operators in the closed string theory. For example the three-point functions determine a product structure. We now discuss the general structure of the algebra of closed string operators α_a at genus 0. There are two types of three-point functions. The most direct one involves just operators transforming as scalars,

$$F_{abc} = \langle \alpha_a \alpha_b \alpha_c \rangle\,. \tag{2.3}$$

We assume that there is a special operator $\mathbb{1}$. Inserting it gives two-point functions $\eta_{ab} = F_{ab0}$, where the index $a = 0$ denotes the special operator. It defines a metric on the space of worldsheet operators. Using the metric, we can raise and lower indices. This allows us to interpret the three-point functions as structure constants for a symmetric product on the space of operators, $\alpha_a \cdot \alpha_b = F^c{}_{ab}\alpha_c$. In this paper we will often denote this product by m. The operator $\mathbb{1}$ serves as a unit for this algebra. We can also construct correlators involving descendants. The natural three-point function is

$$G_{abc} = \left\langle \alpha_a \oint_C \alpha_b^{(1)} \alpha_c \right\rangle , \tag{2.4}$$

where C is a cycle enclosing the insertion point of α_c and not that of α_a. Since we can contract the cycle, this is basically the only three-point function we can construct, except for adding top forms integrated over the worldvolume. It defines the structure constants of a graded antisymmetric product, called the bracket, $\{\alpha_a, \alpha_b\} = \oint_C \alpha_a^{(1)} \alpha_b = G^c{}_{ab}\alpha_c$. We will denote this bracket also by b. It plays an important role in the symmetry algebra of the string theory. Indeed, the first descendant is a current, which acts in this way.

The operations defined by the three-point functions satisfy several well-known relations. These relations, which we will discuss and generalise below, follow from factorisation of the higher correlation functions. First of all, the product m is associative. The bracket b satisfies the Jacobi identity, therefore it is a Lie bracket as expected. The associative product m and the Lie bracket b also satisfy a mutual compatibility, which is similar to the one found for a Poisson algebra. Together, they form an algebra which is thus much like a Poisson algebra. The only difference is that the bracket b has ghost number -1, due to the descendant theory. The resulting structure is called a *Gerstenhaber algebra* (G algebra), see appendix A.[1]

This structure of a G algebra is naturally connected to the topology of 2-dimensional surfaces. If we consider products, we need to insert two operators corresponding to the "in"-state. We start by putting an operator on a point; this corresponds to a puncture in the plane.[2] The topology of this punctured plane Σ_1 remaining for the second operator has two generators: a point, the generator of $H_0(\Sigma_1) = \mathbb{Z}$, and a circle enclosing the puncture, corresponding to $H_1(\Sigma_1) = \mathbb{Z}$. These two generators of the topology naturally correspond to the two bilinear operations in the algebra. The fact that the second nontrivial homology is concentrated in degree one corresponds through the descent equations to the fact that the bracket has degree -1.

The above is precisely the picture arising from the operad of little discs [7]. These are formal structures of discs with holes and gluing relations, which is indeed closely

[1]Gerstenhaber algebras might be more familiar to physicists as substructures of Batalin-Vilkovisky (BV) algebras, after forgetting the BV operator. In fact, in all known cases closed string theories have the full structure of a BV algebra. It is however not known to us at this point that this necessarily should be the case.

[2]The boundary at infinity corresponds to the "out"-state.

related to formal definitions of topological strings. G algebras arise as algebras over the (singular) homology of this operad, and were dubbed 2-algebras in this context.

2.1 Higher correlation functions

Off-shell the Ward identities giving algebraic relations such as the associativity and the Jacobi identity will acquire mild corrections involving the BRST operator. These will involve higher order correlation functions, and the corresponding multilinear operations generate more involved algebraic structures called homotopy algebras. For example, the lowest order correction to the associativity of the product involves a trilinear operation. More generally, the associative algebra becomes a homotopy associative or A_∞ algebra, see e.g. [28]–[32]. Also the Jacobi identity for the bracket will get higher order corrections. The corresponding multilinear maps generate a homotopy Lie or L_∞ algebra, see e.g. [13, 14, 15, 33, 7]. Definitions of A_∞ and L_∞ are given in appendix A, and will also be discussed in more detail later.

As the product and the Lie bracket combine into a Gerstenhaber algebra, it is expected that the A_∞ and L_∞ structure are part of a homotopy Gerstenhaber or G_∞ structure [33, 15]. Several proposals for this structure appeared in the literature. An operad definition of G_∞ was discussed in [34, 16], in connection to (topological) strings. Another version called B_∞ was discussed in [30]. In [35, 8] a related but more general algebraic definition for G_∞ appeared in the context of deformation quantisation and formality. Here we will restrict to a discussion of the A_∞ and L_∞ subalgebras.

Once we take the operators on-shell, the higher correlators remain in general. They still satisfy homotopy algebra relations, though the G algebra decouples in the sense that it will be a genuine subalgebra. This makes it possible to discuss the on-shell description purely in terms of this G algebra structure, as is often done in the literature. However, this is not natural in the context of deformations, as in general the BRST operator deforms, as noted for example, in [36]. Furthermore, higher operations appear naturally in deformation theory, and also turn out to play a crucial role in deformations of string theory, as we will see later, and these appear most naturally off-shell as explained above. It is very hard to to give a complete off-shell definition, and structures are not defined canonically, but rather depend crucially on the insertion points. The definitions we will give below are preliminary, and are strictly speaking only well-defined on-shell.

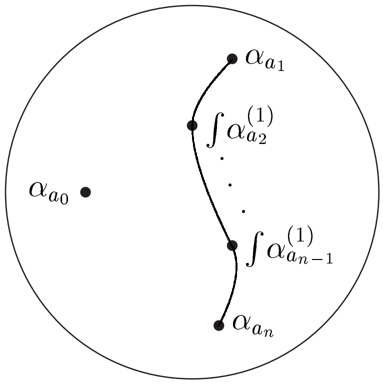

Figure 1: The correlation functions on the sphere defining the A_∞ structure constants. The first descendants are integrated along the indicated path in path order.

The basis of this structure — the BRST operator, the bracket, and the product — was discussed above. The higher structure constants of the L_∞ algebra are defined by the following correlation functions

$$G_{a_0 a_1 \ldots a_n} = \left\langle \alpha_{a_0} \oint \alpha^{(1)}_{a_1} \alpha_{a_2} \int \alpha^{(2)}_{a_3} \cdots \int \alpha^{(2)}_{a_n} \right\rangle =: \left\langle \alpha_{a_0}\, b_n(\alpha_{a_1}, \cdots, \alpha_{a_n}) \right\rangle, \tag{2.5}$$

where the last equality defines the higher multilinear brackets b_n of the L_∞ algebra. It can indeed be shown, using the Ward identities and factorisation, that the corresponding multilinear maps satisfy the L_∞ relations. They are the multilinear string products of [13, 14] expressed in local coordinates on the moduli space (at genus 0), generalised to topological strings by replacing b with G. The proof of the L_∞ relations given in [14] also applies here. Furthermore, the Ward identities for the spin-2 field G assure the graded antisymmetry of these structure constants.

The A_∞ algebra is a bit more involved. As far as we know, the full A_∞ structure has not been studied in the literature, at least we are not aware of any explicit formulas. We propose the following definition of the structure constants for the higher products m_n

$$\begin{aligned} F_{a_0 a_1 \ldots a_n} &= (-1)^{(n-2)g_1 + (n-3)g_2 + \ldots + g_{n-2}} \left\langle \alpha_{a_0} \alpha_{a_1} \int_1^n \alpha^{(1)}_{a_2} \int_2^n \alpha^{(1)}_{a_3} \cdots \int_{n-2}^n \alpha^{(1)}_{a_{n-1}} \alpha_{a_n} \right\rangle \\ &=: \left\langle \alpha_{a_0}\, m_n(\alpha_{a_1}, \cdots, \alpha_{a_n}) \right\rangle, \end{aligned} \tag{2.6}$$

where $g_k = |\alpha_{a_k}|$ denotes the (ghost) grading of the operator α_{a_k}.[3]

They are depicted in figure 1. They involve a chain of path-ordered integrals along a path connecting the insertion points of α_{a_1} and α_{a_n}. These structure constants are not symmetric for $n \geq 3$. They do have however certain symmetry properties due to the Ward identities, at least on-shell. For example, independence of the choice of path on-shell.

Let us motivate this proposal. We start with the trilinear product. The relations in the A_∞ algebra relate this to an off-shell correction to the associativity of the product. Usually, one proves associativity by considering the factorisation of the four-point function $\langle \alpha_a \alpha_b \alpha_c \alpha_d \rangle$ into two three-point functions. Consistency of the factorisation in the s-channel and the t-channel then gives associativity. The factorisation however is corrected once we allow off-shell operators. To find the correct formula, we write the difference of the s-channel and the t-channel factorisation as boundary terms of an integral

$$\left\langle \alpha_{a_0} \alpha_{a_1} \int_1^3 d\alpha_{a_2} \alpha_{a_3} \right\rangle = \left\langle \alpha_{a_0} \alpha_{a_1} \alpha^b \right\rangle \left\langle \alpha_b \alpha_{a_2} \alpha_{a_3} \right\rangle - \left\langle \alpha_{a_0} \alpha^b \alpha_{a_3} \right\rangle \left\langle \alpha_b \alpha_{a_1} \alpha_{a_2} \right\rangle. \tag{2.7}$$

[3]The signs are added because we want to contribute signs coming from the descendants to the operation m_n.

We can use the descent equations to write the total derivative as $d\alpha_{a_2} = Q\alpha_{a_2}^{(1)} + (Q\alpha_{a_2})^{(1)}$, and move the BRST operator in the first term to the other operators. We find the relation

$$\begin{aligned}\left\langle \alpha_{a_0}\alpha_{a_1}\alpha^b\right\rangle\left\langle \alpha_b\alpha_{a_2}\alpha_{a_3}\right\rangle - \left\langle \alpha_{a_0}\alpha^b\alpha_{a_3}\right\rangle\left\langle \alpha_b\alpha_{a_1}\alpha_{a_2}\right\rangle &= \\ = \left\langle \alpha_{a_0}\alpha_{a_1}\int_1^3 (Q\alpha_{a_2})^{(1)}\alpha_{a_3}\right\rangle + (-1)^{g_0+g_1}\left\langle Q\alpha_{a_0}\alpha_{a_1}\int_1^3 \alpha_{a_2}^{(1)}\alpha_{a_3}\right\rangle &+ \\ + (-1)^{g_1}\left\langle \alpha_{a_0}Q\alpha_{a_1}\int_1^3 \alpha_{a_2}^{(1)}\alpha_{a_3}\right\rangle - (-1)^{g_2}\left\langle \alpha_{a_0}\alpha_{a_1}\int_1^3 \alpha_{a_2}^{(1)}Q\alpha_{a_3}\right\rangle &. \end{aligned} \tag{2.8}$$

If all operators are on-shell, this indeed proves associativity. However, off-shell we find corrections from the right-hand side. These corrections have precisely the form of the four-point functions for the A_∞ structure we proposed. We can interpret the factorised correlation functions in terms of compositions of the multilinear maps forming this A_∞ structure. Explicitly, we can write (2.8) as

$$\begin{aligned} m(m(\alpha_{a_1},\alpha_{a_2}),\alpha_{a_3}) - m(\alpha_{a_1}, m(\alpha_{a_2},\alpha_{a_3})) &= \\ = -Q(m_3(\alpha_{a_1},\alpha_{a_2},\alpha_{a_3})) - m_3(Q\alpha_{a_1},\alpha_{a_2},\alpha_{a_3}) &- \\ - (-1)^{g_1}m_3(\alpha_{a_1},Q\alpha_{a_2},\alpha_{a_3}) - (-1)^{g_1+g_2}m_3(\alpha_{a_1},\alpha_{a_2},Q\alpha_{a_3}) &. \end{aligned} \tag{2.9}$$

We can formally write this relation in the form $m\circ m = -Q\circ m_3 - m_3\circ Q$, where $\circ$ is a certain composition of multilinear maps. Note that this composition involves summing over different permutations. The precise definition of this composition will be discussed in more detail later, but can of course be read off from the factorisation in general. This is the correction of the associativity one finds in an A_∞ algebra.

A similar analysis can be performed for the higher products. Though we have not carried out the complete analysis, we will give the general idea of a proof. Commuting a BRST operator through the formula for the higher product m_n, one similarly finds boundary terms. These can be viewed as the chain of $n-2$ ordered integrals being broken up into two chains of length n_1-2 and n_2-2, where $n_1+n_2=n+1$. Note that we need $n_1, n_2 \geq 2$. These boundary terms factorise. This gives a relation of the form

$$Q\circ m_n \pm m_n\circ Q = \sum_{n_1+n_2=n+1} (\pm) m_{n_1}\circ m_{n_2}\,, \tag{2.10}$$

where the signs are determined by the various degrees. This reproduces the A_∞ relations, as will be discussed in more detail later.

3. Deformed correlators and algebraic structure

In this section we discuss deformations of the correlation functions, and therefore of the algebra of the topological closed string theory, by inserting extra closed string operators. The WDVV equations show that these correlators can indeed be interpreted as deformations of the correlation functions. We also discuss how the Gerstenhaber structure of the deforming operators is translated into the Gerstenhaber structure of the multilinear maps.

3.1 WDVV Equations

We can define higher correlators by inserting integrated second descendants. The closed string Ward identities for G assure that these correlators are symmetric in the closed string indices [25, 26]. These relations are known as the WDVV equations. They imply an integrability of the correlation functions: there must exist a function $F(t)$ of formal parameters t^a, such that the higher correlators can be found by differentiating this function. For example, the three-point functions are given by $F_{abc} = \partial_a\partial_b\partial_c F(t)$. Setting $t = 0$ in this relation gives back the original structure constants. However, this equation is valid for nonzero t as well, if we define the deformed three-point functions by formally exponentiating a deformation $\int\alpha^{(2)}$,

$$F_{abc}(t) = \langle \alpha_a\alpha_b\alpha_c \, \mathrm{e}^{t^d\int\alpha_d^{(2)}} \rangle \,, \tag{3.1}$$

where the exponentiated second descendant can be identified with a deformation of the action functional. It shows that indeed the insertions of closed string operators deform the closed string algebra, yielding a deformed A_∞ algebra.

We like to describe the WDVV equations in the context of deformation theory. To facilitate this relation, we will distinguish in the notation between the operators in the algebra and the operators that are used to deform it. We use the notation α_a for the operators in the algebra A we want to deform and ϕ_i for the deforming operators, although for now they are taken from the same algebra.[4] Starting from our proposal (2.6) for the A_∞ algebra, we then write for the deformed higher-point functions

$$\Phi_{ia_0a_1\ldots a_n} = (-1)^{(n-2)g_1+(n-3)g_2+\ldots+g_{n-2}}\Big\langle \alpha_{a_0}\int\phi_i^{(2)}\alpha_{a_1}\int\alpha_{a_2}^{(1)}\cdots\int\alpha_{a_{n-1}}^{(1)}\alpha_{a_n}\Big\rangle \,. \tag{3.2}$$

As for the undeformed A_∞ structure, these definitions are well-defined and path independent if the operators α_a are taken on-shell, but we expect some generalisation off-shell. Upon introducing more deforming operators $\int\phi_j^{(2)}$ etc, the WDVV equations amount to symmetry with respect to all deforming operators.

The reason that we chose the deformations of correlation functions for the A_∞ algebra rather than the correlation functions (2.5) defining the L_∞ algebra is that the latter are not deformed on-shell, as can easily be seen from the Ward identities of G.

We will interpret the $\Phi_{ia_0\ldots a_n}$ in terms of a multilinear map $\Phi_i : A^{\otimes n} \to A$, through the following definition

$$\Phi_{ia_0a_1\ldots a_n} = \langle \alpha_{a_0} \; \Phi_i(\alpha_{a_1},\ldots,\alpha_{a_n})\rangle \,. \tag{3.3}$$

[4] More generally, we could take for the algebras any algebraically closed subalgebras.

These maps are the infinitesimal deformations of the A_∞ algebra structure constants. We will sometimes write $\Phi_i = \Phi(\phi_i)$, to emphasise the relation with the deforming operator. Note that any ϕ_i corresponds to an infinite set of maps, one for any order n.

Let us now examine the deformation of the A_∞ structure more closely. The first-order deformations are simply given by inserting an extra integrated second descendant. Using the Ward identity for G we can also write the corresponding deformed correlator (3.2) as

$$\Phi_{ia_0\dots a_n} = (-1)^{(n-1)g_1+\dots+g_{n-1}}\Big\langle \alpha_{a_0}\phi_i \int \alpha^{(1)}_{a_1}\cdots\int\alpha^{(1)}_{a_n}\Big\rangle. \tag{3.4}$$

The proof is almost the same as the corresponding one for the open string case in [11]. This formula has the advantage that it also applies to the deformation $n \leq 1$. In particular, for $n = 1$, it should give the deformation of the linear map $m_1 = Q$ in the A_∞ algebra. For $n = 1$, we know we can also write the correlation function in the form

$$\Phi_{iab} = \Big\langle \alpha_a \oint \phi_i^{(1)}\alpha_b\Big\rangle. \tag{3.5}$$

This is exactly the deformation of the BRST operator [36].

3.2 Structure of the deformation maps

We want to see the relation between the maps defined by the deformation and abstract deformation theory. To make the notation not too cumbersome, we will assume that the undeformed algebra is a genuine differential associative algebra. That is, the higher products m_n for $n \geq 3$ are zero. For later reference, let us first look at the degree of the operators. For any of the operators α_a we write its ghost number as g_a. The ghost number of its dual operator α^a (with respect to the metric defined by the two-point functions) is written g^a. If Δ is the ghost anomaly, this is given by $g^a = \Delta - g_a$. For the ghost number of the deforming operator we write g_ϕ. When we consider the corresponding map Φ of order n, it has an internal ghost degree given by $g_\Phi = g^{a_0} - \sum_{k=1}^n g_{a_k}$. This ghost number is such that the total ghost number in (3.3) adds up to Δ. Due to ghost number conservation, there is now a relation between the ghost number of the deforming operator ϕ and the ghost number of the corresponding map, given by the corresponding

$$g_\phi = g^{a_0} - \sum_{k=1}^{n}(g_{a_k} - 1) = g_\Phi + n\,. \tag{3.6}$$

The shift in the degree equals the order of the map. This shift is due to the descendants that appear in the correlation functions.

The deforming operators ϕ_i form an algebra, as they are the closed string operators themselves. However, their identification with multilinear maps also gives them an algebraic structure. The algebraic structure of the operators should translate into

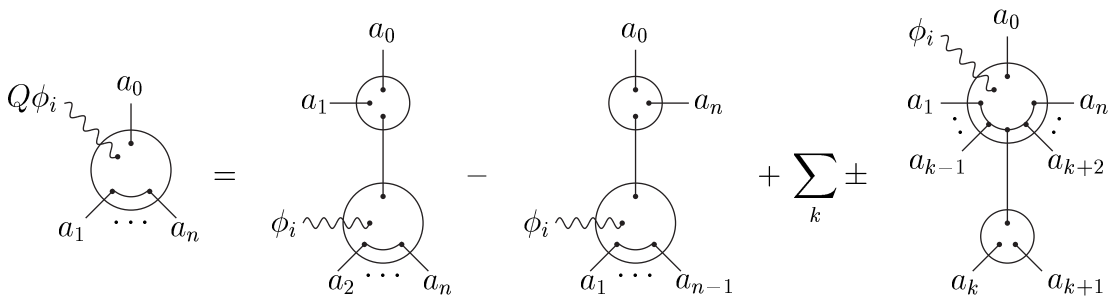

Figure 2: Factorisation expressing the action of the BRST operator.

algebraic operations on these maps. This relation will be crucial in connection with deformation theory.

We start with the action of the BRST operator Q. In order to see this action, we need to consider a deforming operator ϕ not necessarily on-shell. Then there is the following relation

$$\Big\langle \alpha_a \int (Q\phi_i)^{(2)} \alpha_b \alpha_c \Big\rangle = -\Big\langle \alpha_a \oint \phi_i^{(1)} \alpha^e \Big\rangle \Big\langle \alpha_e \alpha_b \alpha_c \Big\rangle + \Big\langle \alpha_a \alpha^e \alpha_c \Big\rangle \Big\langle \alpha_e \oint \phi_i^{(1)} \alpha_b \Big\rangle,$$
$$+(-1)^{(g_i-1)g_b} \Big\langle \alpha_a \alpha_b \alpha^e \Big\rangle \Big\langle \alpha_e \oint \phi_i^{(1)} \alpha_c \Big\rangle, \tag{3.7}$$

where the α operators are taken on-shell. For ϕ on-shell, the left-hand side is zero, and we can interpret the right-hand side as a deformed Leibniz rule. Similarly we find for the four-point function:

$$\begin{aligned}(-1)^{g_1} \Big\langle \alpha_{a_0} \int (Q\phi_i)^{(2)} \alpha_{a_1} \int_1^3 \alpha_{a_2}^{(1)} \alpha_{a_3} \Big\rangle = {} & \Big\langle \alpha_{a_0} \int \phi_i^{(2)} \alpha_{a_1} \alpha^b \Big\rangle \Big\langle \alpha_b \alpha_{a_2} \alpha_{a_3} \Big\rangle - \\ & - \Big\langle \alpha_{a_0} \int \phi_i^{(2)} \alpha^b \alpha_{a_3} \Big\rangle \Big\langle \alpha_b \alpha_{a_1} \alpha_{a_2} \Big\rangle + \\ & + \Big\langle \alpha_{a_0} \alpha^b \alpha_{a_3} \Big\rangle \Big\langle \alpha_b \int \phi_i^{(2)} \alpha_{a_1} \alpha_{a_2} \Big\rangle + \\ & + (-1)^{g_i g_1} \Big\langle \alpha_{a_0} \alpha_{a_1} \alpha^b \Big\rangle \Big\langle \alpha_b \int \phi_i^{(2)} \alpha_{a_2} \alpha_{a_3} \Big\rangle .\end{aligned} \tag{3.8}$$

This equality shows that if ϕ_i is on-shell, that is $Q\phi_i = 0$, the deformed product is associative, at least to first order in the deformation. Thus the BRST operator corresponds to the first-order deformed associator.

There is a generalisation to higher correlators, which is depicted in figure 2. This relation can be stated as

$$\Phi(Q\phi_i) = m \circ \Phi(\phi_i) \pm \Phi(\phi_i) \circ m \,. \tag{3.9}$$

This shows that Q is represented on the algebra of maps on the cohomology by $\Phi \circ m + m \circ \Phi$, where m is the product of the closed string algebra. If the operators

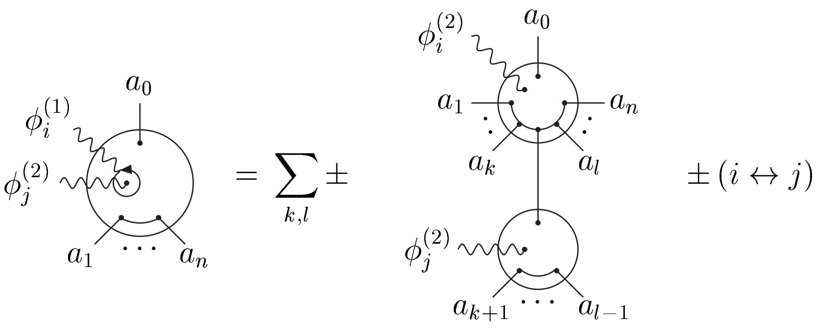

Figure 3: Factorisation for the bracket.

α are not on-shell either, we get corrections from the BRST operator acting on the various α's. $Q\phi$ then corresponds to $Q \circ \Phi_i \mp \Phi_i \circ Q + m \circ \Phi_i \pm \Phi_i \circ m$, where the first terms can be expanded as

$$(Q\circ\Phi_i\mp\Phi_i\circ Q)(\alpha_{a_1},\ldots,\alpha_{a_n}) = Q\Phi_i(\alpha_{a_1},\ldots,\alpha_{a_n})-\sum_i \pm\Phi_i(\alpha_{a_1},\ldots,Q\alpha_{a_i},\ldots,\alpha_{a_n})\,. \tag{3.10}$$

There are also relations between the products and the brackets of deforming operators on the one hand and of factorised correlation functions on the other hand. For these we will be a bit less precise, and only consider the general form. To study them, we have to look at the second-order terms, including two deforming operators. Again we interpret the factorised correlation functions as algebraic operations on the maps Φ_i and Φ_j. For the bracket we study

$$\left\langle \int (Q\phi_i)^{(2)} \int \phi_j^{(2)} \alpha_{a_0}\alpha_{a_1} \int \alpha_{a_2}^{(1)} \cdots \int \alpha_{a_{n-1}}^{(1)} \alpha_{a_n} \right\rangle = 0\,. \tag{3.11}$$

Passing the Q through the descendants gives at one side a boundary term for $\phi_i^{(1)}$ coming close to ϕ_j, which is the map corresponding to $\{\phi_i, \phi_j\}$. Furthermore there are several factorised boundary terms, which have the form

$$\begin{aligned}\left\langle \int \phi_i^{(2)} \alpha_{a_0}\alpha_{a_1} \int \alpha_{a_2}^{(1)} \cdots \int \alpha_{a_{k-1}}^{(1)} \int \alpha_b^{(1)} \int \alpha_{a_{l+1}}^{(1)} \cdots \int \alpha_{a_{n-1}}^{(1)} \alpha_{a_n} \right\rangle \times \\ \times \left\langle \int \phi_j^{(2)} \alpha^b \alpha_{a_j} \int \alpha_{a_{j+1}}^{(1)} \cdots \int \alpha_{a_{l-1}}^{(1)} \alpha_{a_l} \right\rangle,\end{aligned} \tag{3.12}$$

and similar terms with i and j interchanged, as depicted in figure 3. They can be written $\Phi_i \circ \Phi_j \pm \Phi_j \circ \Phi_i$ which can be understood as a supercommutator for higher order maps. This supercommutator therefore corresponds to the deforming operator $\{\phi_i, \phi_j\}$.

Similarly, for the product we have to study the on-shell equality

$$\left\langle \int (Q\phi_i)^{(2)} \int (Q\phi_j)^{(2)} \alpha_{a_0}\alpha_{a_1} \int \alpha_{a_2}^{(1)} \cdots \int \alpha_{a_{n-1}}^{(1)} \alpha_{a_n} \right\rangle = 0\,. \tag{3.13}$$

This one is a bit more involved because it reduces to a codimension 2 boundary. One boundary term now involves the product $\int(\phi_i \cdot \phi_j)^{(2)}$. The other boundary terms are factorisations, defining the product in terms of the maps Φ_i and Φ_j. These boundary terms are of the form

$$\Big\langle \alpha_{a_0}\alpha_b\alpha_c \Big\rangle \Big\langle \int \phi_i^{(2)} \alpha^b \alpha_{a_1} \int \alpha_{a_2}^{(1)} \cdots \int \alpha_{a_{k-2}}^{(1)} \alpha_{a_{k-1}} \Big\rangle \Big\langle \int \phi_j^{(2)} \alpha^c \alpha_{a_k} \int \alpha_{a_{k+1}}^{(1)} \cdots \int \alpha_{a_{n-1}}^{(1)} \alpha_{a_n} \Big\rangle . \tag{3.14}$$

Therefore the map corresponding to the product $\phi_i \cdot \phi_j$ can formally be written in the form $m(\Phi_i, \Phi_j)$.

In conclusion, we found that we could connect to each closed string operator a series of multilinear maps, which can be seen as the deformations of the algebraic structure. Furthermore, we saw that the algebraic structure of the *deforming* closed string – the G algebra formed by Q, m and b – is reflected by a corresponding algebraic structure on the algebra of maps.

4. Hochschild and deformation complexes

In this section we study deformations of closed strings (2-algebras) in a more abstract setting. We saw above that a (topological) closed string theory has the structure of a Gerstenhaber algebra, formed by the BRST operator Q, the OPE product $\cdot$ and the bracket $\{\cdot, \cdot\}$. These are part of an algebra of multilinear maps; this structure will play an essential role in the deformation theory of the closed string algebra. Considering the deformation complex we will find that there can be several different ways to deform this algebra, depending on which part of the structure one wants to deform.

4.1 The Hochschild complex

Mathematically, the deformation of an algebra A is controlled by its Hochschild complex $\mathrm{Hoch}(A)$. Let us first focus on associative algebras A. Operations in A are multilinear maps acting on the vector space A. The vector spaces $C^n(A, A) = \mathrm{Hom}(A^{\otimes n}, A)$, consisting of n-linear maps in A, define the degree n space of what is known in mathematics as the Hochschild complex $\mathrm{Hoch}(A)$ of the algebra A. Algebraic operations and differentials are special elements in this space. Moreover, any deformation of the algebraic structure is naturally an element of the Hochschild complex.

The Hochschild complex of an algebra has an interesting algebraic structure by itself, which plays an important role in the deformation theories. Part of this structure contains information about the algebra A that is deformed. We first extend the action of a map in $\Phi \in C^n(A, A)$ to the full tensor algebra $\mathcal{T}A = \bigoplus_l A^{\otimes l}$. This action is defined as follows

$$\Phi(\alpha_1,\ldots,\alpha_l)=\sum_{k=0}^{l-n}(-1)^{k(n-1)}(\alpha_1,\ldots,\alpha_k,\Phi(\alpha_{k+1},\ldots,\alpha_{k+n}),\alpha_{k+n+1},\ldots,\alpha_l)\,. \quad (4.1)$$

For graded algebras there are extra signs coming from Φ passing the α's. These are standard, and we will not include them in the notation. Through (4.1), we reinterpret the maps in $C^n(A,A)$ as maps on $\mathcal{T}A$, lowering the tensor degree by $n-1$.[5] The composition of multilinear maps is thus a composition on the space $C^*(A,A)$; it is a fundamental operation on the Hochschild complex. The generating action of the composition of two elements $\Phi_i \in C^{n_i}(A,A)$, i.e. its action on $A^{\otimes(n_1+n_2-1)}$, is given by

$$\begin{aligned}(\Phi_1\circ\Phi_2)&(\alpha_1,\ldots,\alpha_{n_1+n_2-1})= \qquad (4.2)\\ &=\sum_{k=0}^{n_1-1}(-1)^{k(n_2-1)}\Phi_1(\alpha_1,\ldots,\alpha_k,\Phi_2(\alpha_{k+1},\ldots,\alpha_{k+n_2}),\alpha_{k+n_2+1},\ldots,\alpha_{n_1+n_2-1})\,.\end{aligned}$$

This definition makes the formulas in the previous section more precise.

Using the composition as a product on $C(A,A)$, we can define a natural supercommutator called the bracket, which is defined by

$$[\Phi_1,\Phi_2]=\Phi_1\circ\Phi_2-(-1)^{(n_1-1)(n_2-1)}\Phi_2\circ\Phi_1\,,\qquad \Phi_i\in C^{n_i}(A,A)\,. \quad (4.3)$$

The order of the map minus one is interpreted as a degree. When A is graded, the maps can also carry an extra grading from this, which would introduce standard extra signs in the definition above. It is easy to show that this bracket satisfies a graded version of the Jacobi identity, making the algebra of maps into a Lie algebra. Notice that the bracket lowers the total order of the maps by one. Because we interpret the order as a degree, the bracket has intrinsic degree -1, so that it is not a regular Lie bracket.

Many familiar relations between algebraic operations can be rewritten elegantly in terms of this structure. The condition on a coboundary operator Q is $Q\circ Q=0$, which can be written in terms of the bracket as $[Q,Q]=0$. The associativity of a bilinear product $m\in C^2(A,A)$ is equivalent to $[m,m]=2m\circ m=0$. The derivation condition of the product (Leibniz rule) can be written $[Q,m]=0$. If we consider a differential associative algebra, with product m and differential Q, these three defining conditions (coboundary, derivation and associativity) can be written as the single equation $[Q+m,Q+m]=0$, by decomposing this into its separate degrees. Notice that although $Q+m$ does not make much sense as a multilinear

[5] We could have started by defining the maps in $C(A,A)$ by their action on the full tensor algebra. It can be shown however that a map on the tensor algebra lowering the degree by $n-1$ is completely determined by its lowest component, that is its action on $A^{\otimes n}$. Hence the definitions are equivalent.

map on A, it does as a map on $\mathcal{T}A$. This definition also makes it almost obvious to introduce A_∞ algebras. An A_∞ algebra is defined in terms of a set of multilinear products $m_n \in C^n(A,A)$, $n = 1, 2, \ldots$, such that m_n has degree $2-n$. These maps should satisfy a generalised associativity condition, which in terms of the total sum $m = m_1 + m_2 + \cdots$ can be written $[m,m] = 0$. By decomposing into the various degrees, this gives an infinite number of relations. For a differential associative algebra, $m_n = 0$ for $n \geq 3$. In general, the first two conditions — coboundary and Leibniz — are not altered. However the associativity condition is changed by the trilinear product m_3 as follows

$$m_2 \circ m_2 + m_1 \circ m_3 + m_3 \circ m_1 = 0\,, \tag{4.4}$$

which is precisely the relation (2.9) we found in the off-shell closed string. On the cohomology with respect to the differential m_1, the product m_2 reduces to an associative product.

Let us assume that we deform a certain bilinear operator $m \in \mathrm{Hom}(A^{\otimes 2}, A)$ (product or bracket), which has an internal degree q, satisfying a certain associativity or Jacobi constraint. Then we can build the following coboundary operator:

$$\delta_m \Phi = m \circ \Phi + (-1)^{n+q|\Phi|} \Phi \circ m = [m, \Phi]\,, \qquad \Phi \in C^n(A,A)\,. \tag{4.5}$$

The coboundary condition $\delta_m^2 = 0$ follows from the associativity of the product m. Also, if we deform a linear operator $m_1 \in \mathrm{End}(A)$ (which should be identified with Q) satisfying a coboundary constraint, it also acts on $C^*(A,A)$ as a coboundary operator. This coboundary operator, defined by $\delta_{m_1} = [m_1, \cdot]$, acts by conjugation on maps in $C(A,A)$. Actually, (4.5) can be applied to any multilinear operation $m_n \in \mathrm{Hom}(A^{\otimes n}, A)$, satisfying a generalisation of the associativity constraint, namely $m_n \circ m_n = 0$. There may be several products that are deformed. The full complex $C^*(A,A)$ then is a multicomplex, having several coboundary operators. For a consistent deformation theory, all these coboundary operators need to commute. This will be guaranteed by additional constraint on the deformed operations (such as Leibniz). We can then construct a total coboundary operator δ, which is a weighed sum of the several coboundary operators.

Any product m defines a cup product $\cup_m$ on the algebra $C^*(A,A)$ by the definition

$$\cup_m(\Phi_1, \Phi_2)(\alpha_1, \ldots, \alpha_{n_1+n_2}) = m\Big(\Phi_1(\alpha_1, \ldots, \alpha_{n_1}), \Phi_2(\alpha_{n_1+1}, \alpha_{n_1+n_2})\Big)\,, \tag{4.6}$$

where $\Phi_i \in C^{n_i}(A,A)$. This can be generalised straightforwardly to general-order products $m_n \in C^n(A,A)$. The corresponding cup product $\cup_{m_n}$ is a product of order n, acting on the algebra $C(A,A)$.

With the coboundary operator and the bracket, the Hochschild complex $C^*(A,A)$ is a (twisted) differential graded Lie algebra. Including the cup product makes it a differential Gerstenhaber algebra.

We can repeat all of the above for the (graded) antisymmetric case, giving a generalisation of Lie algebras. The main difference is that we replace the tensor product $\mathcal{T}A$ by the exterior algebra $\bigwedge^* A$. Maps on the exterior product therefore become antisymmetric multilinear maps on A. The formula for the composition of maps then becomes a signed sum over all permutations of the arguments. For a single bilinear antisymmetric map b, this gives three terms in the formula for $b \circ b$. The vanishing of $b \circ b$ is equivalent to the Jacobi identity. Similarly, one can introduce a coboundary Q, and define a differential Lie algebra by the condition $[Q + b, Q + b] = 0$. More generally, an L_∞ algebra is defined by an infinite number of multilinear antisymmetric maps $b_n : \bigwedge^n A \to A$, $n = 1, 2, \ldots$, of degree $2 - n$ satisfying $[b, b] = 0$, where $b = b_1 + b_2 + \cdots$.

To see the relevance of this structure for deformations, we consider the deformation of an associative product m. We deform the product by an element Φ. The fully deformed product is given by the correlation functions with an exponentiated insertion (3.1). The resulting deformed product is written $m + \Phi$. We want the deformed product to satisfy the generalised associativity condition, i.e. $(m + \Phi)^2 = 0$. Using the associativity of the undeformed product m, we find the condition

$$\delta_m \Phi + \frac{1}{2}[\Phi, \Phi] = 0\,. \tag{4.7}$$

This formula is called the master equation, or Maurer-Cartan equation, of the deformation theory. To first order in the deformation parameter we find that Φ should be closed with respect to the coboundary operator δ_m on the Hochschild complex. We could make this more precise by making a Taylor expansion for the deformation, $\Phi = \sum_{n \geq 1} t^n \Phi_n$, in terms of some deformation parameter t. This gives an infinite number of relations of the form

$$\delta_m \Phi_n = -\frac{1}{2} \sum_{n_1 + n_2 = n} [\Phi_{n_1}, \Phi_{n_2}]\,. \tag{4.8}$$

Note that $n_1, n_2 \geq 1$; therefore this equation can be used to find the higher order corrections to Φ recursively, as Φ_n does not occur on the right-hand side of (4.8). For this, one needs to able to find a left inverse of the operation δ_m, that is to solve the equation $\delta_m \Phi = \Psi$ for Φ, with general Ψ. Any failure for this existence is an obstruction. It means that for a given infinitesimal deformation Φ_1, one may not be able to find higher corrections in order to satisfy the full associativity. Or in other words, not every infinitesimal deformation may be extendable to a full deformation. This can only happen when the third cohomology with respect to δ_m is nonzero, since the right-hand side is in $C^3(A, A)$ and can be shown to be closed with respect to δ_m. From this we see that the second cohomology of δ_m contains the infinitesimal deformations, and the third cohomology contains potential obstructions.

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4.2 Deformation complexes of closed strings

In the previous section we saw a way to deform the algebra of closed strings, by the insertion of extra integrated operators on the worldsheet. In this section we discuss deformations in the context of the deformation complex, which describes the basic cohomology theory governing the deformation of the algebra. The deformation complex $\mathrm{Def}(A)$ of an algebra is a graded Lie algebra containing all possible deformations of this algebra. In any deformation theory of algebras, the central role in the deformation complex is played by the Hochschild complex, which we already met. The grading is such that $\mathrm{Def}^1(A)$ corresponds to the infinitesimal deformations of A, $\mathrm{Def}^0(A)$ contains the (global) symmetries, and $\mathrm{Def}^2(A)$ contains potential obstructions to extend the infinitesimal deformations to finite ones. The other gradings correspond to higher symmetries and higher obstructions. Generally, the deformation complex can be decomposed (as a vector space) as $\mathrm{Def}(A) = A \oplus C(A, A)$. The first factor A is quite trivial, and corresponds to shifts of the elements of the algebra (translations in A). The second factor $C(A, A)$ is the Hochschild complex, containing deformations of the products. In the following we will ignore the first factor, as it will play no significant role in the discussion. The most important effect of this factor is that it kills the first factor $C^0(A, A) = A$ in the Hochschild cohomology, corresponding to maps of order 0.[6]

Up to now, we have treated the algebra A merely as a vector space, and we did not yet use any information about the product structure it may have. This information will supply the vector space $C(A, A)$ with some extra structure. Most important for the deformation theory is the fact that there will be a coboundary operator δ on $C(A, A)$, making it into a complex. This coboundary operator will precisely be determined by the algebraic structure that is deformed. Indeed, we saw earlier that the deformation of a bilinear product m satisfying an associativity condition defines a coboundary operator δ_m on $C(A, A)$. Moreover, we saw that this coboundary operator was closely related to the deformation problem of the product m: the cohomology of appropriate degree describes the possible infinitesimal deformations. This shows precisely what we need in addition to define the deformation complex. We need a coboundary operator δ of degree 1 with respect to an appropriate grading on the space $C(A, A)$. This coboundary operator is determined by the structure we are deforming. The grading also plays an important role. It determines how the different maps in the deformation complex should be interpreted; for example, the true deformations have degree 1, the elements of degree 0 are related to symmetries, and the degree 2 elements describe obstructions. Indeed, we know from examples in physics that the interpretation of several operations or operators can depend on the deformation problem one studies.

[6]As $C(A, A)$ corresponds to deformations of the background, and the first factor A can be interpreted as a perturbation of the theory by the operators in the theory itself, we speculate that this cancellation should be interpreted physically as background independence.

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If A is merely a complex — a graded vector space with a coboundary operator Q of degree 1, it can be considered a 0-algebra (the state space of a point). In this case the only thing we can deform is Q, so we should take for the coboundary on the deformation complex the operator δ_Q. The deformation complex now has the structure of a differential associative algebra. The product is given by the composition $\circ$, which obviously is associative. Actually, the only relevant part turns out to be $A \oplus \mathrm{End}(A)$, forming the algebra of affine transformations on A [7]. Hence the cohomology of the deformation complex of a 0-algebra is an associative algebra, or a 1-algebra in the language of [7].

If A is an associative algebra, we deform the product m. Then the coboundary operator on the deformation complex is given by δ_m. This deformation complex has the structure of a Gerstenhaber algebra, formed by the coboundary δ_m, the cup product $\cup_m$, and the Gerstenhaber bracket. Hence the deformation complex of a 1-algebra is a 2-algebra. If A is a differential associative algebra, we can also deform its coboundary operator Q. This would supply the deformation complex with a second coboundary operator δ_Q. The natural question then arises as to which one of the two coboundary operators defines the structure of complex for the deformation complex. The answer is both. To see how this works notice that in this situation the vector space $C(A, A)$ has a double grading. One grading comes from the map degree, which we denote n, the other comes from the internal grading of A, let's call it q. The space of maps break up into doubly graded spaces $\mathrm{Hom}(A^{\otimes n}, A)^q$. The two coboundary operators δ_Q and δ_m have bidegrees (q, n) given by $(1, 0)$ and $(0, 1)$ respectively, and make $C(A, A)$ into a double complex. The essential condition that the two coboundary operators have to anticommute follows from the Leibniz rule. The total coboundary operator of the double complex is given by the sum. This also implies that the total degree on the complex is given by the sum, $p = q + n - 1$, so that both coboundary operators raise the total degree by 1. Here the shift by 1 is related to a mathematical convention, which requires degree $p = 0$ in the deformation complex to correspond to symmetries, and we definitely want $\mathrm{End}(A)^0$ (reparametrisations) to be interpreted as symmetries of the algebra. Also, this fits nicely with the structure of the Hochschild complex, in which $n - 1$ turns up as a natural grading.

With all this in mind, we now turn to the deformation of a 2-algebra. We will be working mainly on-shell, therefore the algebraic structure is that of a differential Gerstenhaber algebra, as was considered in the beginning of section 2. As the BRST operator in general is deformed we also need to include it in our discussion. We know that off-shell structure should be a homotopy algebra, but we will assume that we can work in this restricted setting.[7] In this situation, there are three operators which we can potentially deform. This gives us three different coboundary operators,

[7]Using operad descriptions of the more general off-shell structure one can in principle define a more general deformation theory for these. However this becomes much more involved, and will be far beyond the scope of this paper.

δ_Q, δ_m, δ_b, on the space $C(A,A)$. Corresponding to these coboundaries, there are three types of arrows in the complex. The diagonal ones come from the product, the vertical ones from the BRST operator, and the horizontal ones from the bracket. The deformation complex therefore looks as follows.

$$\begin{array}{ccccccc}
A^0 & \overset{\delta_b}{\rightarrow} & \mathrm{End}(A)^{-1} & \rightarrow & \mathrm{Hom}(A^{\otimes 2},A)^{-2} & \rightarrow & \cdots \\
\delta_Q\downarrow & \searrow^{\delta_m} & \downarrow & \searrow & \downarrow & \searrow & \\
A^1 & \rightarrow & \mathrm{End}(A)^{0} & \rightarrow & \mathrm{Hom}(A^{\otimes 2},A)^{-1} & \rightarrow & \cdots \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
A^2 & \rightarrow & \mathrm{End}(A)^{1} & \rightarrow & \mathrm{Hom}(A^{\otimes 2},A)^{0} & \rightarrow & \cdots \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
A^3 & \rightarrow & \mathrm{End}(A)^{2} & \rightarrow & \mathrm{Hom}(A^{\otimes 2},A)^{1} & \rightarrow & \cdots \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow &
\end{array} \tag{4.9}$$

The natural thing to do now is to define a total coboundary which is basically the sum of the three. However, it is impossible to define a degree on the complex such that all three maps have degree 1. Physically, this means that we cannot for example identify gauge symmetries and true deformations for all three operators simultaneously. This implies that we cannot consistently deform all three operators at the same time. What can happen physically is that deformations of one operator (corresponding to deformations of degree 1 in the deformation complex) are obstructions for deformations of another operator (having degree larger than 1 in that deformation complex), or otherwise (gauge) symmetries (having degree smaller than 1).

It is possible to deform two structures at the same time. Then we keep two of the three arrows in the complex, and we find a double complex. First note, that we can introduce a pair of quantum numbers, (p,q) say, such that one of the maps has quantum numbers $(1,0)$, and the other $(0,1)$. For example, if we keep the vertical and horizontal arrows, we can take the row and column numbers. The total degree then is simply the sum of the two, so that both maps indeed have degree 1. The total differential is more or less the sum of the maps (up to some relative signs). The degrees should always be chosen such that the space $\mathrm{End}(A)^0$ has degree $(0,0)$, as indeed these should certainly be interpreted as gauge symmetries. From the point of view of the algebra, we are really deforming only a substructure. The three substructures we can deform correspond to the differential associative (DA) structure, the differential Lie (DL) structure, and the Gerstenhaber (G) structure.

5. Classification of closed string deformations

In this section we will discuss the three possibilities for deforming the structure of a closed string algebra separately.

The deformation complex breaks down into the vector spaces $\mathrm{Hom}(A^{\otimes n},A)^q$, where n is the order and q denotes the internal ghost degree (that is, a corresponding

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Algebra	Total coboundary	Total degree p
DA	$\delta_m + \delta_Q$	$q + n - 1$
DL	$\delta_b + \delta_Q$	$q + 2n - 2$
G	$\delta_b + \delta_m$	$n - 1$

Table 1: The three deformations of a 2-algebra: differential algebra, differential Lie algebra and Gerstenhaber algebra. The last column gives the formula for the total degree p in the deformation complex, in terms of the internal ghost number q and the order n.

map raises the internal degree in A by q). The operations m we want to deform are particular elements in this space, so they also carry the corresponding degrees. It is easily seen that if m has ghost degree q and order n, than the corresponding coboundary operator δ_m increases the ghost degree by q and the order by $n-1$. From this now we can derive the expression of the total degree in the deformation complex. The necessary condition is that for each operator m that is deformed, the corresponding coboundary operator δ_m should have total degree 1. Here we use the degrees (q, n) of the various operations: δ_Q has degrees $(1, 0)$, δ_m has degree $(0, 1)$, and δ_b has degrees $(-1, 1)$. The various possibilities for choosing the degree in such a way that two operations are deformed are given in table 1. The offset of the degree is determined by the fact that obviously $\mathrm{End}(A)^0 \subset \mathrm{Def}^0(A)$.

Before we turn to a description of the three cases, we will first examine the significance of the degrees.

5.1 Gradings and dimensions

In general the gradings have the form $p = \alpha q + \beta(n-1)$, the most general linear relation between the degrees such that $\mathrm{End}(A)^0$ corresponds to degree $p = 0$. We have to be careful with the definition of these degrees. In general, the degrees p and q refer to the ghost numbers of the zeroth descendants, modulo a shift in the definition of p. For example, for the bracket $\{\alpha_a, \alpha_b\}$ the degree is given by $q = g_{\{\alpha_a,\alpha_b\}} - g_{\alpha_a} - g_{\alpha_b} = -1$.

We want to argue here that the coefficient β is related to the dimensionality of the deforming theory. To see this, let us look at more general topological field theories in any dimension d. They always come with a Lie bracket, which is the generalisation of the bracket in two dimensions, and is defined by

$$\{\phi_1, \phi_2\} = \oint_C \phi_1^{(d-1)} \phi_2 \,, \tag{5.1}$$

where C is a $(d-1)$-cycle enclosing the insertion point of ϕ_2 (a $(d-1)$-sphere). For $d = 1$ this gives the commutator with respect to the product, as the cycle C consists of the (formal) difference of two points. This Lie bracket has degree $1 - d$, due to the descendant. Restricted to the BRST-closed operators, it is easily seen to be independent of the choice of the cycle C and to satisfy the Jacobi identity when restricted to the BRST-closed operators.

There is a natural relation between this Lie bracket and the quantum commutator in canonical quantisation. In a canonical quantisation, we use a time slicing for our space-time, and a time-coordinate x^0. Assume two canonically quantised operators $\hat{\phi}_1$ and $\hat{\phi}_2$ satisfying a commutation relation of the form

$$[\hat{\phi}_1^{(d-1)}(y), \hat{\phi}_2(x)] = \hat{\phi}_3(x)\delta(y-x)\,. \tag{5.2}$$

Here the $(d-1)$th descendant is natural, because the delta function should be considered a $(d-1)$-form. If we want to calculate the Lie bracket defined above in a canonical quantisation, we should split the cycle C up into two half-spheres $C = D_+ \cup D_-$, at times $y^0 > x^0$ and $y^0 < x^0$, according to the time-slicing we chose. We can deform these half-spheres to two space-slices, pushing the strip on the side to infinity, where it should not give any contribution. The quantisation of the bracket than can be written

$$\{\hat{\phi}_1, \hat{\phi}_2\}(x) = \int_{D_+} \hat{\phi}_1^{(d-1)}(y)\hat{\phi}_2(x) - \int_{D_-} \hat{\phi}_1^{(d-1)}(y)\hat{\phi}_2(x) = \int_D \hat{\phi}_3(x)\delta(y-x) = \hat{\phi}_3(x)\,, \tag{5.3}$$

where we used the quantum commutator above. Hence we see that indeed the quantum commutator directly maps to the Lie bracket. This procedure is well-known in the context of two-dimensional CFT's, where it describes the action of currents. In a canonical quantisation we can therefore relate the operator ϕ_1 to a differential operator $\phi_3 \frac{d}{d\phi_2}$. Because of the descendant in the definition of the bracket the operator ϕ_1 and the corresponding map differ in degree by an amount of $d-1$. This indeed corresponds to $\beta = d-1$.

If we consider a pair of canonically conjugate operators $\phi_2 = \phi$ and $\phi_1 = \pi$, we have $\phi_3 = 1$. Let us for simplicity work in a first-order formalism, where π and ϕ are both fundamental fields. In this case it is straightforward to connect to the Hochschild complex. The canonical quantisation gives $\pi^{(d-1)} \sim \frac{d}{d\phi}$, which shows that the operator π corresponds to an element of $\mathrm{Hom}(A, A)$ in the Hochschild complex. More generally, the operator π^n gives an element in $\mathrm{Hom}(A^{\otimes n}, A)$. Let us compare the various degrees. The degrees refer explicitly to the degrees of the zeroth descendants of the operators. Except for the degree p in the deformation complex, which in d dimensions is shifted by $d-1$. Denote the ghost degrees of ϕ and π as g_ϕ and g_π respectively. In the action there should be a term of the form $\int \pi^{(d-1)} d\phi$ as we are working in a first-order formalism. This implies that $g_\pi = -g_\phi + d - 1$. An operator of the form π^n (and its descendants) now corresponds to an element in the deformation complex of degree $p = ng_\pi - d + 1$. The induced multilinear map has a component in the maps of degree n which acts as $(d/d\phi)^{\otimes n}$, which has an explicit ghost number $q = -ng_\phi$. Comparing the two degrees we find

$$p = ng_\pi - d + 1 = n(-g_\phi + d - 1) - d + 1 = q + (d-1)(n-1)\,. \tag{5.4}$$

More generally, the operator π^n induces other multilinear maps of the form $\pi^m \times (d/d\phi)^{\otimes(n-m)}$, which are in $\mathrm{Hom}(A^{\otimes(n-m)}, A)^q$, where $q = mg_\pi - ng_\phi$. These maps can be considered as different descendants of the operator π^n. Comparing the degrees one finds exactly the same relation, if we replace n by the order $n-m$ of the map. We conclude that the coefficient β equals $d-1$.

Let us comment on the shift in degree in the deformation complex. This has to do with the mathematical convention for the degree in the deformation complex. This is such that the actual deformations have degree 1. These operators should however correspond to the physical operators, which in d dimensions have ghost number d. This is because the corresponding perturbation of the action, $\int \phi^{(d)}$, should have ghost number 0. Therefore, we have to shift the degree by $d-1$, $p = g_\phi - d + 1$. The mathematical degree can be considered the degree of the $(d-1)$th descendant of the operator, which defines a perturbation of a "pre-Lagrangian" $\widetilde{L}$, which is defined such that $S = \int dt\, \widetilde{L}^{(1)}$.

5.2 Deformation of the differential associative structure

Of the three possibilities, we first consider the deformation of the DA structure (differential algebra, or more generally the A_∞ structure). The DA structure is determined by the symmetric product and the BRST operator Q. In physics this is the best-known problem, and in the context of topological strings it gives rise to the WDVV equations [26]. It is basically the problem of deforming the closed string using closed string operators. There is also a deformation of the BRST operator Q, which was studied in this context already [36]. The two structures together define the bracket in the usual way. But as is known, deformations of the closed string by closed string operators do not deform the bracket. Therefore, we expect that in general the bracket will be fixed and is not deformed.

The deformation double complex for the deformation of the DA structure has the following structure

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
A^0 & \xrightarrow{\delta_m} & \mathrm{End}(A)^0 & \to & \mathrm{Hom}(A^{\otimes 2}, A)^0 & \to & \cdots \\
{\scriptstyle\delta_Q}\downarrow & & \downarrow & & \downarrow & & \\
A^1 & \to & \mathrm{End}(A)^1 & \to & \mathrm{Hom}(A^{\otimes 2}, A)^1 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A^2 & \to & \mathrm{End}(A)^2 & \to & \mathrm{Hom}(A^{\otimes 2}, A)^2 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{5.5}$$

The vertical arrows correspond to the coboundary δ_Q, while the horizontal arrows correspond to δ_m. So we see that the two gradings have a very natural interpretation: one (related to the BRST operator) is the internal ghost degree (target space degree), and the other (related to the product) is the map degree of the multilinear maps (the number of elements in the algebra on which it acts).

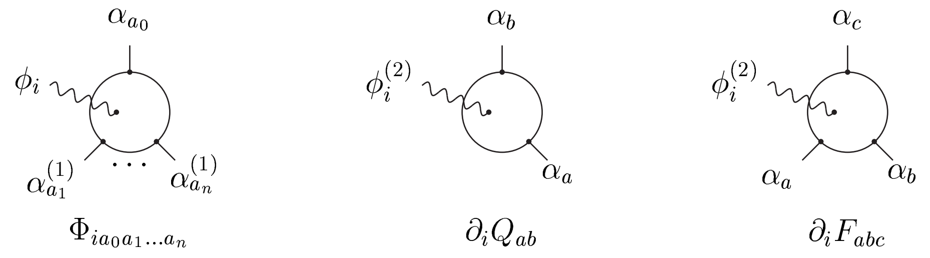

Figure 4: A typical diagram corresponding to an element of the Hochschild cohomology for a deformation of the associative structure, the three-point function giving the deformation of the BRST operator, and the four-point function giving the deformation of the product.

The degree in the deformation complex is given by $p = n - 1 + q$. This means that the degree p cocycles in the deformation complex are given by the elements of the space

$$\mathrm{Def}^p(A) = \bigoplus_{n\geq 0} \mathrm{Hom}(A^{\otimes n}, A)^{p-n+1} \,. \tag{5.6}$$

The most important part of the deformation complex is the degree 1 space. These contain the actual deformations of the algebra. This space is given by

$$\mathrm{Def}^1(A) = A^2 \oplus \mathrm{End}(A)^1 \oplus \mathrm{Hom}(A^{\otimes 2}, A)^0 \oplus \mathrm{Hom}(A^{\otimes 3}, A)^{-1} \oplus \cdots \,. \tag{5.7}$$

This is very natural from the point of view of the string algebra. The terms in the physical deformations contain the deformed operations. For example the deformation of the BRST operator is an element of $\mathrm{End}(A)^1$, and the deformation of the product is an element of $\mathrm{Hom}(A^{\otimes 2}, A)^0$.

The formula for the total degree (5.4) suggests that we should take the first descendants for the n incoming closed string operators in a string diagram corresponding to $\Phi_{ia_0\dots a_n}$. This is indicated in figure 4.

This is precisely the structure we found in the case of WDVV. Also, the formula for the degrees matches up exactly with the one for WDVV, (3.6), if we take into account the remarks of the last subsection: the definition of the degree of the map matches exactly, $q = g_\Phi$, while the ghost degree of ϕ is shifted by $1 = d - 1$, so that $p = g_\phi - 1$.

The Hochschild cohomology has the structure of a Gerstenhaber algebra, with the bracket having degree -1. We saw that for deformation of the closed string by itself this algebra could be identified with the (on-shell) algebra of the closed string itself.

We conclude that the WDVV equations describe a deformation theory of the DA structure.

5.3 Deformation of the differential Lie structure

Secondly, we consider the deformation of the differential Lie algebra structure, formed by the BRST operator Q and the bracket. Now we find for the deformation complex the following form.

$$
\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
A^0 & \xrightarrow{\delta_b} & \mathrm{End}(A)^{-1} & \rightarrow & \mathrm{Hom}(A^{\otimes 2}, A)^{-2} & \rightarrow & \cdots \\
{\scriptstyle \delta_Q}\downarrow & & \downarrow & & \downarrow & & \\
A^1 & \rightarrow & \mathrm{End}(A)^{0} & \rightarrow & \mathrm{Hom}(A^{\otimes 2}, A)^{-1} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A^2 & \rightarrow & \mathrm{End}(A)^{1} & \rightarrow & \mathrm{Hom}(A^{\otimes 2}, A)^{0} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A^3 & \rightarrow & \mathrm{End}(A)^{2} & \rightarrow & \mathrm{Hom}(A^{\otimes 2}, A)^{1} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & &
\end{array}
\tag{5.8}
$$

The vertical arrows are again determined by δ_Q; the horizontal arrows correspond to δ_b in this case. The total degree in the deformation complex is given by $p = 2n-2-q$. The actual deformations, that is the deformations of degree one, are given by

$$
\mathrm{Def}^1(A) = A^3 \oplus \mathrm{End}(A)^1 \oplus \mathrm{Hom}(A^{\otimes 2}, A)^{-1} \oplus \mathrm{Hom}(A^{\otimes 3}, A)^{-3} \oplus \cdots . \tag{5.9}
$$

The terms in the deformation complex at degree one show which operations are potentially deformed. We find the maps of degree one, corresponding to deformations of the BRST operator, and bilinear maps of degree -1, indicating the deformation of the bracket. The next term, that is trilinear maps of degree -3, will also play an important role in the deformation theory, as we will see below. This is the deformation complex that is most natural from the mathematical point of view, and in the mathematics literature it is referred to as describing the deformations of a Gerstenhaber algebra [7, 8]. The Hochschild cohomology has the structure of a Poisson algebra, as also found in [7]. The degree of the Poisson bracket, which is given by (4.3), has degree -2, which is even.

Following (5.4), we expect that this deformation theory should be considered as a 3-dimensional theory. The shift by $2n-2$ is typical for a 3-dimensional theory. We will argue in the next section that indeed this deformation theory enters naturally in the topological open membrane. The shift by $2n$ also indicates that for the higher correlation functions corresponding to the deformation complex, the extra insertions come as first descendants, as suggested in figure 5.

5.4 Deformation of the Gerstenhaber structure

Lastly, we consider the deformation of the Gerstenhaber structure, consisting of the product and the bracket. The deformation double complex now has the following form

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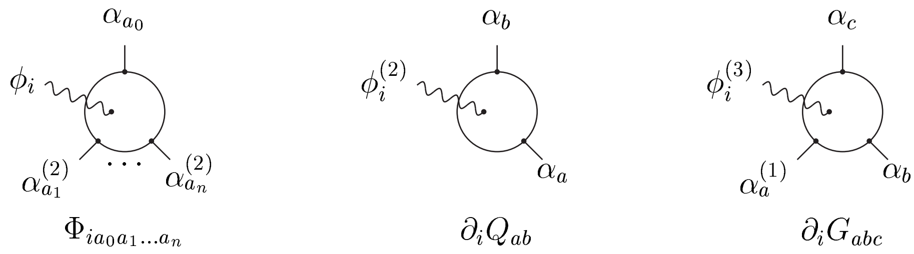

Figure 5: A typical diagram corresponding to an element of the Hochschild cohomology for a deformation of the differential Lie structure, the three-point function giving the deformation of the BRST operator, and the four-point function giving the deformation of the bracket.

$$\begin{array}{ccccccccc}
 & & & & & & \downarrow & & \\
 & & & & A^0 & \xrightarrow{\delta_b} & \mathrm{End}(A)^{-1} & \to & \cdots \\
 & & & & \delta_m\downarrow & & \downarrow & & \\
 & & A^1 & \to & \mathrm{End}(A)^0 & \to & \mathrm{Hom}(A^{\otimes 2},A)^{-1} & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
A^2 & \to & \mathrm{End}(A)^1 & \to & \mathrm{Hom}(A^{\otimes 2},A)^0 & \to & \mathrm{Hom}(A^{\otimes 3},A)^{-1} & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{End}(A)^2 & \to & \mathrm{Hom}(A^{\otimes 2},A)^1 & \to & \mathrm{Hom}(A^{\otimes 3},A)^0 & \to & \mathrm{Hom}(A^{\otimes 4},A)^{-1} & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{5.10}$$

Now the arrows correspond to δ_m and δ_b respectively. The total degree is the degree of the map (the number of elements on which it acts), modulo a shift. The internal degree in the algebra A does not contribute to the degree of the deformation complex.

The degree in the deformation complex is given by $p = n - 1$. The gauge symmetries and matter content are given by the zeroth and first degree deformations, respectively:

$$\mathrm{Def}^0(A) = \mathrm{End}(A), \qquad \mathrm{Def}^1(A) = \mathrm{Hom}(A^{\otimes 2}, A)\,. \tag{5.11}$$

The grading is the same as for the Hochschild complex, apart from a shift by one. This implies that the bracket has degree -2, and the Hochschild cohomology has the structure of a Poisson algebra. The ghost degree does not play any role in the deformation theory. Therefore, we expect that the original ghost number symmetry is broken in this case.

6. Topological open membranes and boundary strings

In [11], we studied deformations of boundary theories for open strings by bulk operators. We found that the deformation theory of this 1-algebra indeed had the structure of a 2-algebra. This would lead us to expect that the 3-algebra deformation of the 2-algebra formed by the closed strings can be found in the context of open membranes. In this section we will argue that this is indeed the case.

To see this we interpret the closed strings as the boundary theory of a topological open membrane, or TOM for short. The relevant algebra will be the algebra of boundary operators, and has the structure of a 2-algebra. Indeed, this is a closed string theory. The BRST operator Q of the open membrane descends to this boundary string, by integrating the corresponding current over a half-sphere enclosing the boundary operator. The deforming algebra, formed by the ϕ_i, is the bulk algebra of the membrane.

6.1 Three-dimensional topological field theories

Three-dimensional topological field theories can be treated in a manner quite similar to two-dimensional ones, so we will be quite brief here. There are three-point functions C_{ijk} defining a symmetric product, which are equivalent to the two-dimensional ones. Using a unit operator we define a metric by the two-point function equivalent to C_{0ij}. The bracket is now defined by the three-point functions

$$B_{ijk} = \left\langle \phi_i \oint \phi_j^{(2)} \phi_k \right\rangle , \tag{6.1}$$

where we integrate over a 2-sphere enclosing ϕ_k.

As for any TFT, we demand the presence of a BRST operator Q and of an operator G, such that $\{Q, G\} = d$. In the presence of a boundary, these operators also induce an action on the boundary operators, though in general there may be extra boundary terms. The symmetry current G in the topological open membrane induces a Ward identity of the form

$$\begin{aligned} 0 = &\sum_m \xi^\mu(x_m) \Big\langle \prod_n \phi_{i_n}(z_n) \alpha_{a_1}(x_1) \cdots G_\mu \alpha_{a_m}(x_m) \cdots \alpha_{a_r}(x_r) \Big\rangle + \\ &+ \sum_n \xi^\mu(z_n) \Big\langle \phi_{i_1}(z_1) \cdots G_\mu \phi_{i_n}(z_n) \cdots \phi_{i_s}(z_s) \prod_m \alpha_{a_m}(x_m) \Big\rangle, \end{aligned} \tag{6.2}$$

where the z's are points in the bulk and the x's are points on the boundary. Here the operators ϕ and α can be any operator, not necessarily BRST-closed. They can also be descendants. In this equation, ξ^μ is a globally defined conformal vector field. The conformal group of the 3-ball is $\mathrm{SO}(2, 2)$, which is six-dimensional. Therefore, we have a basis of 6 vector fields to choose for the ξ's. This counting relies very much on a conformally invariant gauge fixing of the open membrane. A priori we do not know if such a gauge fixing does exist. In the following we will assume this, rather than considering it as a requirement of the TOM.

6.2 Deformations

We study deformations of the closed string correlation functions by including new operators ϕ_i in the correlation functions, which we view as deforming operators.

However, these operators will now be bulk operators for the membrane. We can define mixed two-point functions by

$$\Phi_{ia} = \langle \phi_i \alpha_a \rangle \, . \tag{6.3}$$

The mixed three-point functions are defined by

$$\Phi_{iab} = \left\langle \phi_i \alpha_a \int \alpha_b^{(2)} \right\rangle . \tag{6.4}$$

Notice that we cannot have any correlators "in between"; if we would insert a first descendant, integrated over a cycle, we could always shrink the cycle to zero. Higher mixed correlators are given by

$$\Phi_{ia_0a_1\ldots a_n} = \left\langle \phi_i \alpha_{a_0} \int \alpha_{a_1}^{(2)} \cdots \int \alpha_{a_n}^{(2)} \right\rangle . \tag{6.5}$$

We will assume that the closed string Ward identities for G are still valid, so that these correlators are symmetric in the closed string indices. For the relevant situations, we will argue below that this is indeed the case.

When we introduce extra membrane operators in the Φ's, we should integrate them,

$$\Phi_{ija_0a_1\ldots a_n} = \left\langle \phi_i \int \phi_j^{(3)} \alpha_{a_0} \int \alpha_{a_1}^{(2)} \cdots \int \alpha_{a_n}^{(2)} \right\rangle . \tag{6.6}$$

Now the algebra of deforming operators is assumed to have the same structure as the closed string theory. That is, we have Q and G. Also, these operators should be related to the corresponding operators on the closed string theory. This would mean that the correlators are also symmetric in the i, j indices. This should also be true if we introduce extra integrated deforming operators. Indeed, the G operator is zero on these top forms. These assumptions imply that the mixed correlators are integrable: there are functions $\Phi_{a_0\ldots a_n}(t)$ such that $\Phi_{ia_0\ldots a_n}(t) = \partial_i \Phi_{a_0\ldots a_n}(t)$, where $\partial_i = \frac{\partial}{\partial t^i}$. The coefficients in the expansion in t are the higher correlation functions. We can therefore formally write these deformed correlators as

$$\Phi_{a_0a_1\ldots a_n}(t) = \left\langle \alpha_{a_0} \int \alpha_{a_1}^{(1)} \alpha_{a_2} \int \alpha_{a_3}^{(2)} \cdots \int \alpha_{a_n}^{(2)} \, \mathrm{e}^{t^i \int \phi_i^{(3)}} \right\rangle . \tag{6.7}$$

6.3 The algebraic structure of open membranes

The essential identity needed to view the insertions of bulk operators as a deformation of the boundary algebra was the symmetry of the higher correlators $\Phi_{ija_0a_1\ldots a_n}$ defined in (6.6), with respect to the bulk indices:

$$\left\langle \phi_i \phi_j^{(3)} \alpha_a \right\rangle = C \left\langle \phi_i^{(3)} \phi_j \alpha_a \right\rangle , \tag{6.8}$$

where C should be a function of the insertion points. In order for the integrated correlation functions to be truly invariant under this switch, this function should be

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the Jacobian of the coordinate transformation from the insertion point of ϕ_i to the insertion point of ϕ_j.[8] We will now argue that the assumption of conformal invariance gives enough global Ward identities to give the above relation at least. We are however not in a position to determine the factor C, due to a lack of understanding of the conformal invariance. Therefore the invariance of the integrated correlation functions will not be established completely. As we argued, assuming conformal invariance we have 6 independent Ward identities of the form (6.2). However, in the present case we do not want the boundary operator α_a to get involved. This can be established if the vector field ξ^μ used in the Ward identity is 0 at the insertion point of this operator. This gives two restrictions on ξ, leaving us with 4 Ward identities. These are however just sufficient to transfer the 3 independent components of G_μ from ϕ_j to ϕ_i, thereby establishing the existence of the above relation. As G is 0 on any second descendant of a boundary operator or a third descendant of a bulk operator, the relation remains true if we insert any number of these maximal descendants.

More important is a relation of the form

$$\left\langle \int \phi_i^{(3)} \alpha_a \oint \alpha_b^{(1)} \alpha_c \right\rangle = \left\langle \phi_i \int \alpha_a^{(2)} \int \alpha_b^{(2)} \alpha_c \right\rangle , \tag{6.9}$$

showing that we can interpret the mixed correlation functions as bulk to boundary metrics deformed by the boundary operators. This can be proved using Ward identities of the form

$$\left\langle \phi_i^{(3)} \alpha_a \alpha_b^{(1)} \alpha_c \right\rangle = C \left\langle \phi_i \alpha_a^{(2)} \alpha_b^{(2)} \alpha_c \right\rangle . \tag{6.10}$$

We start from the right-hand side. A priori, we have 6 independent global vector fields. Next we choose the vector fields that fix the position of α_c. As this gives two conditions, there are 4 vector fields. Of these, we use two vector fields to transfer the second descendant from α_b to ϕ_i. Next we choose the third vector field such that it fixes the position of α_b as well (as this gives two more conditions, there are two independent choices). We can use this to transfer one descendant of α_a to ϕ_i without getting additional terms. This argument shows that conformal invariance of the TOM theory is large enough to get this Ward identity (in fact, we only need 5 independent vector fields). Again, we cannot decide whether C is a Jacobian. We expect this to be true on the general basis of conformal invariance and will assume it henceforth. Equation (6.9) means that the correlator Φ_{iabc} is a deformation of the bracket. It would remain valid when we include extra fully integrated bulk and boundary insertions.

We want to view the mixed correlators as intertwiners between the closed membrane algebra and the deformations of the G_∞ structure, given by the boundary correlators $G_{abc...}$. An essential structure of the topological bulk theory is the BRST

[8] Conformal invariance guarantees the existence of this coordinate transformation.

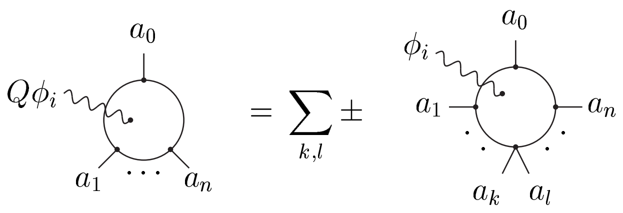

Figure 6: Factorisation of the BRST operator.

operator. A BRST operator acting on the closed string operator in the mixed correlators can be deformed to a contour around the boundary operators. Using the descent equations for the boundary operators gives the following identity, also depicted in figure 6.

$$\begin{aligned}\Big\langle Q\phi_i\alpha_{a_0}\int\alpha^{(2)}_{a_1}\cdots\int\alpha^{(2)}_{a_n}\Big\rangle &= \Big\langle\phi_i\{\alpha_{a_0},\alpha_{a_1}\}\int\alpha^{(2)}_{a_2}\cdots\int\alpha^{(2)}_{a_n}\Big\rangle+\\ &\quad+(-1)^{n+1}\Big\langle\phi_i\int\alpha^{(2)}_{a_1}\cdots\int\alpha^{(2)}_{a_{n-1}}\{\alpha_{a_n},\alpha_{a_0}\}\Big\rangle+\\ &\quad+\sum_{k=1}^{n-1}(-1)^k\Big\langle\phi_i\alpha_{a_0}\int\alpha^{(2)}_{a_1}\cdots\int\{\alpha_{a_k},\alpha_{a_{k+1}}\}^{(2)}\cdots\int\alpha^{(2)}_{a_n}\Big\rangle.\end{aligned} \tag{6.11}$$

In this derivation, the boundary operators are taken on-shell (BRST-closed), while for ϕ_i we take an arbitrary local membrane operator. The boundary terms in the factorised diagrams are related to points in the moduli space where two boundary operators approach each other. They arise from a total derivatives of the form $\int d\alpha^{(1)}_a = \int Q\alpha^{(2)}_a$. Its boundary term near another boundary operator will still contain a first descendant, which is integrated around the insertion point. Thus it involves the bracket rather than the product. We find that the bulk BRST operator corresponds to the operator δ_b. More generally, if we include off-shell boundary operators we find that there are corrections from the boundary BRST operator, which are easily seen to correspond to the coboundary δ_Q acting on the maps. The coboundary operator on the deformation complex is therefore found to be $\delta_Q+\delta_b$, which is indeed the coboundary operator related to deformations of the DL (or more generally of the L_∞) structure.

We can do the same with the inclusion of a second bulk operator, that is we look at the factorisation of the correlation function

$$\Big\langle\int(Q\phi_i)^{(3)}\int\phi^{(3)}_j\alpha_{a_0}\oint\alpha^{(1)}_{a_1}\alpha_{a_2}\int\alpha^{(2)}_{a_3}\cdots\int\alpha^{(2)}_{a_n}\Big\rangle, \tag{6.12}$$

which vanishes on-shell. The basic difference is that the undeformed products m (of any order) are replaced by the deformed products. Furthermore, there is an extra boundary term related to the two bulk operators coming close together. This involves

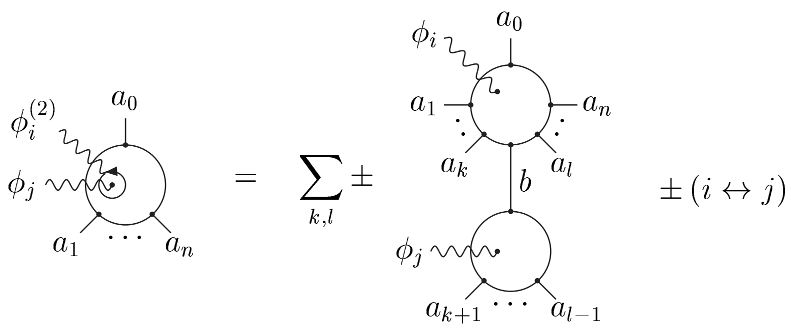

Figure 7: Factorisation of the bracket.

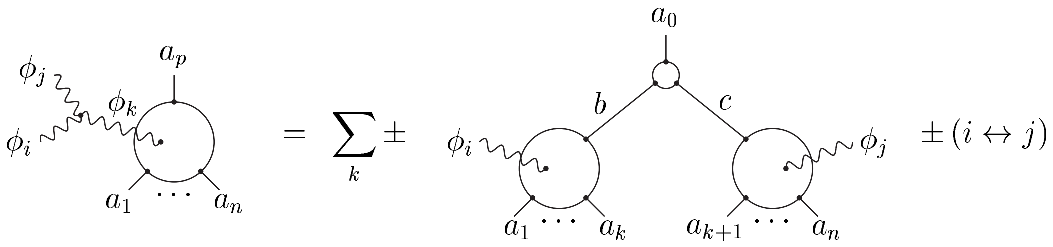

Figure 8: Factorisation of the product.

the integral of the second descendant of ϕ_i around ϕ_j, because of the total derivative term $d\phi_i^{(2)}$ coming from pulling Q through the descendants. It gives the bracket in the membrane theory. The vanishing of (6.12) gives the relation depicted in figure 7,

$$\Phi(\{\phi_i, \phi_j\}) = [\Phi(\phi_i), \Phi(\phi_j)]\,. \tag{6.13}$$

There is also a factorisation giving the bulk product as a boundary term, and several factorised correlation functions as the other boundary terms. However, it involves a codimension 3 boundary, starting from the deformed correlator with two deforming operators. This can be seen from the fact that we need to replace $\int \phi_i^{(3)} \int \phi_j^{(3)}$ by a single descendant $\int (\phi_i \cdot \phi_j)^{(3)}$. The factorisation is depicted in figure 8. From the fact that we have a codimension 3 boundary, it can be seen that the undeformed factor involves the bracket of the boundary theory. This gives the following identity:

$$\Phi(\phi_i \cdot \phi_j)(\alpha_{a_1}, \ldots, \alpha_{a_n}) = \sum_k \pm \{\Phi_i(\alpha_{a_1}, \ldots, \alpha_{a_k}), \Phi_j(\alpha_{a_{k+1}}, \ldots, \alpha_{a_n})\}\,. \tag{6.14}$$

7. The topological open membrane

In this section we discuss as an example an explicit topological open membrane theory. The model we will study is the membrane with only a WZ term, whose action is given by

$$S = \int_M \frac{1}{6} C_{ijk} dX^i \wedge dX^j \wedge dX^k\,. \tag{7.1}$$

This action appears for example as a suitable decoupling limit of the open supermembrane in M-theory [18]. This action as it stands is quite singular for calculating correlation functions, as it is cubic. In order to allow ourselves to do calculations and quantise the action, we use a first-order formalism and BV quantisation techniques, as developed in [27]. In this section we will only state the main points of the calculation and the final results, as it is just intended as a first example of the nontrivial deformation of the DG structure. More worked-out calculations will appear in a forthcoming paper of one of the authors [37].

7.1 BV quantisation of the topological open membrane

The explicit topological open membrane we will study is a BV-quantised membrane theory, which was discussed in [27]. This theory is very much inspired by the Cataneo and Felder model (CF) for the topological open string with a B-field WZ term [10]. The easiest way to write down the CF model is to use superfields; these are functions on the worldsheet of bosonic coordinates x^μ and fermionic coordinates θ^μ. These superfields combine all the fields: physical fields, ghost fields and antifields. In the CF model there were two sets of superfields, which we will denote here[9] $\boldsymbol{X}^i(x,\theta)$ and $\boldsymbol{\chi}_i(x,\theta)$ — the first one bosonic, the second fermionic. They are the generating functionals of the scalars X^i and χ_i and their descendants. This formalism can be viewed as a quantisation of the open string: the boundary operators are functions of the superfields $\boldsymbol{X}^i$, while the superfields $\boldsymbol{\chi}_i$ play the role of "momenta". Moreover, together they generate the Hochschild cohomology of the open string algebra, e.g. $\boldsymbol{\chi}_i$ represents $\frac{\partial}{\partial \boldsymbol{X}^i} \in \mathrm{Hom}(A,A)$.

The explicit BV quantisation of the TOM theory defined by the WZ term goes very much along the same lines. We will not give an elaborate motivation, as this goes outside the scope of the present paper. Instead we will simply pose the model here, and give motivation for it later, by showing that the undeformed TOM is equivalent to the topological closed string theory given by CF. From the philosophy above, in order to construct the TOM we have to introduce two more sets of superfields, which we denote $\boldsymbol{\psi}^i$ and $\boldsymbol{F}_i$, which serve as "momenta" for the two superfields $\boldsymbol{X}^i$ and $\boldsymbol{\chi}_i$. The four superfields describing the TOM can be expanded as

$$
\begin{aligned}
\boldsymbol{X}^i &= X^i + \rho^i_\mu \theta^\mu + \frac{1}{2} X^i_{\mu\nu}\theta^\mu\theta^\nu + \frac{1}{6}\rho^i_{\mu\nu\lambda}\theta^\mu\theta^\nu\theta^\lambda, \\
\boldsymbol{\chi}_i &= \chi_i + H_{i\mu}\theta^\mu + \frac{1}{2}\chi_{i\mu\nu}\theta^\mu\theta^\nu + \frac{1}{6}H_{i\mu\nu\lambda}\theta^\mu\theta^\nu\theta^\lambda, \\
\boldsymbol{\psi}^i &= \psi^i + A^i_\mu\theta^\mu + \frac{1}{2}\psi^i_{\mu\nu}\theta^\mu\theta^\nu + \frac{1}{6}A^i_{\mu\nu\lambda}\theta^\mu\theta^\nu\theta^\lambda, \\
\boldsymbol{F}_i &= F_i + \eta_{i\mu}\theta^\mu + \frac{1}{2}F_{i\mu\nu}\theta^\mu\theta^\nu + \frac{1}{6}\eta_{i\mu\nu\lambda}\theta^\mu\theta^\nu\theta^\lambda.
\end{aligned}
$$

[9] These superfields were called $\tilde{X}$ and $\tilde{\eta}$ in [10].

These fields have ghost degree 0, 1, 1, and 2, respectively. The scalar components $(X^i, \chi_i, \psi^i, F_i)$ can be viewed as coordinates on the superspace $\Pi T(\Pi T^* M)$. Here Π is an operator that shifts the degree in the fibre by one. Viewing (x^μ, θ^μ) as coordinates on the superspace ΠTN, where N is the worldvolume of the membrane, these fields can be viewed as parametrising a map $\Pi TN \to \Pi T(\Pi T^* M)$ between the two superspaces. We will choose boundary conditions such that the new fields $\boldsymbol{\psi}^i$ and $\boldsymbol{F}_i$ vanish on ∂N. This means that the boundary ∂N maps to the base space $\Pi T^* M$ of the target space.

In order to get a BV quantisation, we need to introduce a BV (anti)bracket. From our motivation of choosing $\boldsymbol{F}$ and $\boldsymbol{\psi}$ as the "momenta" of the superfields $\boldsymbol{X}$ and $\boldsymbol{\chi}$, we have a natural symplectic structure on the superfields above,

$$\boldsymbol{\omega}_{BV} = \int_N \int d^3\theta \Big(\delta \boldsymbol{X}^i \delta \boldsymbol{F}_i + \delta \boldsymbol{\chi}_i \delta \boldsymbol{\psi}^i \Big) , \tag{7.2}$$

where δ denotes the d-operator (De Rham differential) on field space. This is a symplectic form of ghost degree -1. This symplectic structure defines the BV bracket, which is dual to it, and can formally be written

$$(\cdot,\cdot)_{BV} = \frac{\partial}{\partial \boldsymbol{X}^i} \wedge \frac{\partial}{\partial \boldsymbol{F}_i} + \frac{\partial}{\partial \boldsymbol{\chi}_i} \wedge \frac{\partial}{\partial \boldsymbol{\psi}^i} . \tag{7.3}$$

This is easily seen to derive from a BV operator $\triangle$.

Motivated by CF, we will write down an undeformed membrane theory using a Poisson bivector b^{ij} on M, i.e. $b^{il}\partial_l b^{jk} + perms. = 0$. The BV action we propose is given by

$$S_0 = \int_N \int d^3\theta \Big(\boldsymbol{F}_i D \boldsymbol{X}^i + \boldsymbol{\psi}^i D \boldsymbol{\chi}_i + \boldsymbol{F}_i \boldsymbol{\psi}^i + \boldsymbol{b}^{ij} \boldsymbol{F}_i \boldsymbol{\chi}_j + \frac{1}{2} \partial_k \boldsymbol{b}^{ij} \boldsymbol{\psi}^k \boldsymbol{\chi}_i \boldsymbol{\chi}_j \Big) , \tag{7.4}$$

where $D = \theta^\mu \partial_\mu$, and $\boldsymbol{b}^{ij}$ denotes the pull-back by the superfield $\boldsymbol{X}$ to ΠTN, $\boldsymbol{b}^{ij}(x,\theta) = b^{ij}(\boldsymbol{X}(x,\theta))$. It is easily seen that the BV action above satisfies both the classical and the quantum master equation, $\triangle S_0 = (S_0, S_0)_{BV} = 0$. The BRST operator is determined through $\boldsymbol{Q}_0 = (S_0, \cdot)_{BV}$. Because the auxiliary fields $\boldsymbol{F}_i$ appear only linearly, we can exactly integrate them out. After solving for $\boldsymbol{\psi}_i$ in this equation, the action reduces to a pure boundary term

$$S_{CF} = \int_{\partial N} \int d^2\theta \Big(\boldsymbol{\chi}_i D \boldsymbol{X}^i + \frac{1}{2} \boldsymbol{b}^{ij} \boldsymbol{\chi}_i \boldsymbol{\chi}_j \Big) = \int_{\partial N} \Big(H_i dX^i + \chi_i d\rho^i + \frac{1}{2} b^{ij} H_i H_j + \cdots \Big) . \tag{7.5}$$

This is precisely the action of the Cattaneo-Felder model, as announced. This is related to the usual topological closed string with just the B-field WZ term by integrating out H.

The boundary operators are determined by functions f of the scalar fields X^i and χ_i, that is functions on the base space $\Pi T^* M$ of the target space. The corresponding boundary operator α_f and its descendants combined as

$$\alpha_f + \theta\alpha_f^{(1)} + \frac{1}{2}\theta^2\alpha_f^{(2)} = f(\boldsymbol{X}, \boldsymbol{\chi}) . \tag{7.6}$$

We will sometimes denote this by $\boldsymbol{f}$. It is natural to view the space of functions on $\Pi T^* M$ as the polynomial algebra $\mathbb{C}[\{X^i\}, \{\chi_i\}]$ generated by X^i and χ_i. By formally replacing the fermionic generators χ_i by the basic vector fields ∂_i, one sees that the boundary operators are in one-to-one correspondence with the multi-vector fields on the target space M. Our undeformed boundary algebra A will thus be the algebra of multi-vector fields $A = \Gamma(M, \bigwedge^* TM)$.

The three-point functions determine a structure of an algebra on these boundary operators, which indeed turns out to be a 2-algebra. More precisely, for the bracket this relation will be given by

$$\Big\langle \alpha_\delta \alpha_f \alpha_g \Big\rangle \equiv \Big\langle \alpha_\delta \alpha_{f\cdot g} \Big\rangle, \qquad \Big\langle \alpha_\delta \oint \alpha_f^{(1)} \alpha_g \Big\rangle \equiv \Big\langle \alpha_\delta \alpha_{\{f,g\}} \Big\rangle , \tag{7.7}$$

where we took the outgoing state corresponding to a δ-function on the target space. The product is easily seen to be the wedge product on the multi-vector fields. The bracket is given by

$$\{f, g\} = \frac{\partial f}{\partial \chi_i}\frac{\partial g}{\partial X^i} \pm \frac{\partial f}{\partial X^i}\frac{\partial g}{\partial \chi_i} . \tag{7.8}$$

This is a consequence of the $\boldsymbol{\psi}^i \boldsymbol{F}_i$ term in the BV action. As a bracket on the multi-vector fields, this bracket is well-known in mathematics. It is called the Schouten-Nijenhuis bracket. It is the unique extension to the full algebra of multi-vector fields of the Lie bracket on vector fields.

When b is nonzero, there is also a differential, so that the boundary string theory is a differential Gerstenhaber algebra. This differential is given by the BRST operator restricted to the boundary,

$$Qf = b^{ij}\chi_j \frac{\partial f}{\partial X^i} + \frac{1}{2}\partial_i b^{jk}\chi_j\chi_k \frac{\partial f}{\partial \chi_i} . \tag{7.9}$$

It is easily checked that Q is nilpotent if b^{ij} is a Poisson structure.

We conclude that the undeformed topological open membrane we proposed above is given by the algebra of multi-vector fields. It is supplied with the differential Q above, the wedge product and the Schouten-Nijenhuis bracket, which indeed makes it into a 2-algebra. Our next task is to study the deformation of this 2-algebra. We first propose a natural deformation in the context of our BV-quantised theory.

7.2 Deformations of the TOM

The boundary string theory will be deformed by coupling the TOM to a bulk operator. We can construct bulk operators corresponding to functions $f(X, \chi, \psi, F)$

on the full target space $\Pi T(\Pi T^*M)$. They are given by the pull back to the world-volume $\boldsymbol{f} = f(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\psi}, \boldsymbol{F})$, using the superfields. This generates all descendants of the operator and to conserve ghost number, this should have degree 3. The natural topological deformation is to turn on a 3-form deformation in the open membrane theory. Given a 3-form c this defines an operator ϕ_c, which for $b = 0$ is given by

$$\int_N \phi_c^{(3)} = \int_N \int d^3\theta \, \frac{1}{6} \boldsymbol{c}_{ijk} \boldsymbol{\psi}^i \boldsymbol{\psi}^j \boldsymbol{\psi}^k \,. \tag{7.10}$$

We will use this operator as the deformation of the BV action functional. It will turn out that the b-field in general does not have to define a strict Poisson structure in the deformed case, so we will for now drop this requirement. The totally deformed BV action, including b, becomes [27]

$$\begin{aligned} S = \int_N \int d^3\theta \Big(& \boldsymbol{F}_i D\boldsymbol{X}^i + \boldsymbol{\psi}^i D\boldsymbol{\chi}_i + \boldsymbol{F}_i \boldsymbol{\psi}^i + \boldsymbol{b}^{ij} \boldsymbol{F}_i \boldsymbol{\chi}_j + \frac{1}{2} \partial_k \boldsymbol{b}^{ij} \boldsymbol{\psi}^k \boldsymbol{\chi}_i \boldsymbol{\chi}_j + \\ & + \frac{1}{2} \boldsymbol{b}^{il} \partial_l \boldsymbol{b}^{jk} \boldsymbol{\chi}_i \boldsymbol{\chi}_j \boldsymbol{\chi}_k + \frac{1}{6} \boldsymbol{c}_{ijk} (\boldsymbol{\psi}^i + \boldsymbol{b}^{il} \boldsymbol{\chi}_l)(\boldsymbol{\psi}^j + \boldsymbol{b}^{jm} \boldsymbol{\chi}_m)(\boldsymbol{\psi}^k + \boldsymbol{b}^{kn} \boldsymbol{\chi}_n) \Big) . \end{aligned} \tag{7.11}$$

This action functional satisfies the BV master equation if the total field strength given by

$$\boldsymbol{h}^{ijk} = \boldsymbol{b}^{il} \partial_l \boldsymbol{b}^{jk} + \boldsymbol{b}^{jl} \partial_l \boldsymbol{b}^{ki} + \boldsymbol{b}^{kl} \partial_l \boldsymbol{b}^{ij} + \boldsymbol{b}^{il} \boldsymbol{b}^{jm} \boldsymbol{b}^{kn} \boldsymbol{c}_{lmn} \,, \tag{7.12}$$

vanishes. Notice that this implies that b^{ij} is not necessarily a Poisson structure.

If we now integrate out $\boldsymbol{F}$, the second line in (7.11) reduces to the WZ term (7.1) of the c-field. This motivates our choice for the deforming operator, and for the whole model, since it shows that the model serves as a well-defined quantum action for the ill-defined theory based on the WZ term.

To calculate the first-order corrections to the algebraic structure we need to calculate the corresponding correlation functions, which define the map Φ_c corresponding to the operator ϕ_c. This can be related to a deformation on the algebra of multi-vector fields. For example, we can write

$$\Phi_c(\alpha_f, \alpha_g) \equiv \alpha_{\{f,g\}_1} \,, \tag{7.13}$$

where $\{\cdot,\cdot\}_1$ is the first-order deformation of the bracket on the multi-vector fields. In the next subsection we will use the Hochschild complex to calculate the effect on the algebra, at least in a first-order quantisation. The field theory we now have can in principle be used to calculate the correspondence of the Hochschild cohomology — the deforming operators — and the Hochschild complex — the differential operators — as a perturbation series in c (formality).

7.3 Hochschild cohomology of the 2-algebra of multi-vector fields

In section 4, we saw that the possible deformers are essentially given by elements of the Hochschild cohomology. We will now calculate this cohomology for the topological open membrane theory. We start with the situation $b = 0$.

We saw that the operators of the boundary closed string form the algebra of functions on ΠT^*M, which we represent by the algebra of polynomials $A = \mathbb{C}[\{X^i\},\{\chi_i\}]$. As explained above, this corresponds to the algebra of multi-vector fields $\Gamma(M,\bigwedge^* TM)$. This is naturally a graded algebra, with the degree corresponding to the vector degree. This means that the generators X^i have degree 0, and χ_i have degree 1. This algebra indeed has the structure of a Gerstenhaber algebra or 2-algebra, with the product m given by the wedge product and the bracket b given by the Schouten-Nijenhuis bracket, defined by

$$\{\alpha,\beta\} = \frac{\partial\alpha}{\partial X^i}\frac{\partial\beta}{\partial\chi_i} + (-1)^{|\alpha|}\frac{\partial\alpha}{\partial\chi_i}\frac{\partial\beta}{\partial X^i}\,. \tag{7.14}$$

This is about the simplest nontrivial 2-algebra one can construct.

The deformation of the Gerstenhaber algebra of multi-vector fields is determined by the Hochschild cohomology. The Hochschild complex is given by the algebra of multilinear operators acting on the algebra A, $\mathrm{Hoch}(A) = \bigoplus_n \mathrm{Hom}(A^{\otimes n}, A)$. This is the algebra of multi-differential operators. On the cohomology act several differentials, mainly related to the product structures on the algebra A. Taking the partial cohomology with respect to the differential δ_m associated to the ordinary product, we can describe these multi-differential operators by introducing anticommuting coordinates ψ^i, representing ∂_{χ_i}, and commuting variables F_i, representing ∂_{X^i}. The Hochschild cohomology can be described as a polynomial algebra: $H^*_{\delta_m}(\mathrm{Hoch}(A)) = \mathbb{C}[\{X^i\},\{\chi_i\},\{\psi^i\},\{F_i\}]$, see appendix B. The degree of the generators ψ^i is 1, while the degree of F_i should be taken 2.[10] There is still a differential left, related to the bracket. It is defined in a similar way to the Gerstenhaber differential, but with the product replaced by the bracket. This differential is easily calculated on the above polynomial algebra to be given by

$$\delta_b = \psi^i\frac{\partial}{\partial X^i} + F_i\frac{\partial}{\partial\chi_i}\,, \tag{7.15}$$

which correctly has degree 1. The full Hochschild cohomology is now the cohomology of the above polynomial algebra with respect to this differential. This algebra has a natural Poisson structure of degree -2, given by

$$\{\alpha,\beta\} = \frac{\partial\alpha}{\partial X^i}\frac{\partial\beta}{\partial F_i} - (-1)^{|\alpha|}\frac{\partial\alpha}{\partial\chi_i}\frac{\partial\beta}{\partial\psi^i} \pm (\alpha\leftrightarrow\beta)\,. \tag{7.16}$$

The structure of the differential δ_b, the bracket of degree -2 and the product makes the Hochschild cohomology into a 3-algebra, which is just a differential Poisson algebra, except from the degree of the bracket.

The cohomology with respect to the differential δ_b removes all dependence on χ and F, so that in the end we are left with only polynomials of X^i and ψ^i. Hence

[10] These correspond precisely to the extra fields in the BV action. This correspondence can in fact be taken quite seriously.

the cohomology equals that of the differential forms on $\mathbb{R}^n$. Note that the Poisson bracket of the 3-algebra is identically zero on the cohomology.

In general, for $A = \Gamma(M, \bigwedge^* TM)$, we find $H^* \operatorname{Hoch}(A) = H^*(M)$. This means that for sufficiently large p, we have $H^p(\mathrm{Def}(A)) = H^{p+2}(M)$. Especially, $H^1(\mathrm{Def}(A)) = H^3(M)$. This term in the complex determines the actual deformations. The element in the Hochschild cohomology corresponding to a closed 3-form c is represented by the polynomial $\frac{1}{6}c_{ijk}(X)\psi^i\psi^j\psi^k$. We are of course interested in the corresponding element in the full Hochschild complex, that is the map Φ_c deforming the algebra. To find it, remember that ψ^i corresponds to the operator $\frac{\partial}{\partial\chi_i}$ in the complex. This corresponds to a naive canonical quantisation, which gives

$$\frac{1}{6}c_{ijk}(X)\frac{\partial}{\partial\chi_i} \wedge \frac{\partial}{\partial\chi_j} \wedge \frac{\partial}{\partial\chi_k}\,. \tag{7.17}$$

Of course, this is only the leading term in the map from the Hochschild cohomology to the complex.[11] Notice that this is a trilinear differential operator. This means that a trilinear product in the L_∞ algebra is deformed.

Let us now turn on a b-field b^{ij}, which we will take constant for simplicity. This introduces a derivation Q on the algebra, and the calculation of the cohomology for the double complex is more complicated, as we now have two coboundary operators δ_Q and δ_b on the complex. The total coboundary operator on the double complex $C = \operatorname{Hoch}(A)$ is given by $D = d + \delta = \delta_b + \delta_Q$. With both differentials nonzero, we can in general calculate the cohomology using spectral sequence techniques, see appendix C. This basically amounts to solving a series of descent equations. Starting from a class d-closed element α_0, we have descent equations $\delta\alpha_0 = -d\alpha_1$, etcetera. The two coboundary operators on the double complex are given by

$$\delta \equiv \delta_Q = b^{ij}\chi_j\frac{\partial}{\partial X^i} + b^{ij}F_j\frac{\partial}{\partial\psi^i}, \qquad d \equiv \delta_b = \psi^i\frac{\partial}{\partial X^i} + F_i\frac{\partial}{\partial\chi_i}\,. \tag{7.18}$$

It turns out that the descent equations can be solved introducing the following operator

$$\gamma = b^{ij}\chi_j\frac{\partial}{\partial\psi^i}\,. \tag{7.19}$$

It is easily checked that $[d, \gamma] = -\delta$. This can be used to solve $\alpha_1 = \gamma\alpha_0$, $\alpha_2 = \gamma\alpha_1$, and so on.

Let us see what this implies for the deformation term, when we turn b on. First note that the operator d is not effected by turning on b, therefore we still conclude that the d-cohomology class α_0 is represented by an element $\frac{1}{p!}\alpha_{i_1\ldots i_p}(X)\psi^{i_1}\cdots\psi^{i_p}$, where $\alpha_{i_1\ldots i_p}(X)$ is a closed p-form. The effect of γ is to replace ψ^i by $b^{ij}\chi_j$. Therefore,

[11] It should be compared to the leading term $\theta^{ij}\partial_i \wedge \partial_j$ for the deformation of the product in noncommutative geometry.

the total class α is given in terms of the same form, but with ψ^i replaced by $\psi^i + b^{ij}\chi_j$,

$$\alpha = \frac{1}{p!}\alpha_{i_1\ldots i_p}(X)(\psi^{i_1} + b^{i_1 j_1}\chi_{j_1})\cdots(\psi^{i_p} + b^{i_p j_p}\chi_{j_p})\,. \tag{7.20}$$

Most interestingly, the class in the third cohomology related to the closed 3-form c is given by

$$\frac{1}{6}c_{ijk}\psi^i\psi^j\psi^k + \frac{1}{2}c_{ijk}b^{il}\chi_l\psi^j\psi^k + \frac{1}{2}c_{ijk}b^{il}b^{jm}\chi_l\chi_m\psi^k + \frac{1}{6}c_{ijk}b^{il}b^{jm}b^{kn}\chi_l\chi_m\chi_n\,. \tag{7.21}$$

This corresponds precisely to the deformation term in the action (7.11).

Using the first-order map (the "quantisation") from the cohomology to the Hochschild complex, this translates into the following set of deformed operations in the algebra:

$$\begin{aligned} Q &= b^{ij}\chi_j\frac{\partial}{\partial X^i} + \frac{1}{2}(\partial_k b^{ij} + c_{klm}b^{li}b^{mj})\chi_i\chi_j\frac{\partial}{\partial\chi_k} + \mathcal{O}(c^2)\,, \\ \{\cdot,\cdot\} &= \frac{\partial}{\partial X^i}\wedge\frac{\partial}{\partial\chi_i} + \frac{1}{2}c_{ijk}b^{kl}\chi_l\frac{\partial}{\partial\chi_i}\wedge\frac{\partial}{\partial\chi_j} + \mathcal{O}(c^2)\,, \\ \{\cdot,\cdot,\cdot\} \equiv b_3 &= \frac{1}{6}c_{ijk}\frac{\partial}{\partial\chi_i}\wedge\frac{\partial}{\partial\chi_j}\wedge\frac{\partial}{\partial\chi_k} + \mathcal{O}(c^2)\,. \end{aligned} \tag{7.22}$$

The corrections to the BRST operator Q and the bracket $\{\cdot,\cdot\}$ are precisely given by the corrections in the higher terms of the spectral sequence: the sum is simply the quantisation of the total representative. These operations satisfy the relations of a "L_3 algebra". Together with the undeformed product, it satisfies the relations of a "G_3 algebra".

More precisely, the above operations should be calculated by computations of the corresponding membrane correlators. Indeed, direct tree-level computations confirm the naive quantisation rules [37] to this order in c. More generally, higher order corrections to these operations can be given by loop calculations in the TOM.

7.4 Effective target space action

We will comment briefly on the consequence of the deformations we found.

The correlation functions determine an effective action in the target space M, which is defined as the generating functional of the correlation functions of the boundary operators. As we saw, the boundary operators are related to functions of X and χ, which can be identified with multi-vector fields. The physical fields in the effective action correspond to the physical boundary operators in the open membrane theory. These are the operators of ghost degree 2 $\mathcal{B} = \frac{1}{2}B^{ij}(X)\chi_i\chi_j$, which correspond to degree 2 multi-vector fields. Interpreting the effective action as the generating functional of the correlation functions $F_{a_0\ldots a_n}$ of the open membrane theory gives in general an effective action functional which to first order in c can be written in the form

$$S_{\text{eff}} = \int_{\Pi T^* M} \left(\frac{1}{2} \mathcal{B} \cdot Q\mathcal{B} + \frac{1}{3} \mathcal{B} \cdot \{\mathcal{B}, \mathcal{B}\} + \frac{1}{4} \mathcal{B} \cdot \{\mathcal{B}, \mathcal{B}, \mathcal{B}\} \right) , \tag{7.23}$$

where we integrate over the zero-modes of X^i and χ_i. Precisely such a form for the action of the closed string field theory was proposed by Zwiebach [14] for the bosonic closed string, which was shown to satisfy the (quantum) master equation. Generalising his proposal for more general closed string field theories, this is of course what it reduces to in the case of the TOM. The integration over χ picks out the top component in terms of the multi-vector degree, which is nonzero only for $D = 5$. In other dimensions, we cannot consistently truncate to the physical degrees of freedom, and we also have to take into account other non-physical modes. It seems that 5 dimensions is very natural for this action. In this situation the action is an interacting topological field theory which is very reminiscent of Chern-Simons, but with a 2-form gauge field. This is closely related to the way the Chern-Simons action arises in topological open string theory [38], which is exactly the analogue for the open string derivation we gave here for the open membrane. Notice that this theory is already interacting for $c = 0$, as we still have a cubic term coming from the bracket. The c-field gives a further quartic interaction term.

We can indeed interpret much of the deformation theory in terms of a generalised gauge theory. Let us first go to a representation in terms of differential forms rather than multi-vector fields. This can be done if we take as a background an invertible b^{ij}, and write the algebra A in terms of $\chi^i = b^{ij}\chi_j$. Indeed functions of X^i and χ^i can be identified with differential forms, if we identify $\chi^i = dX^i$. In this identification, the BRST operator Q, for $c = 0$, is identified with the De Rham differential.

Turning on a boundary operator $B = \frac{1}{2} B^{ij} \chi_i \chi_j$ effects Q. The perturbed BRST operator has the form

$$Q_B = Q + \{B, \cdot\} , \tag{7.24}$$

in terms of the bracket on the algebra of multi-vector fields. This can be interpreted as a covariant d-operator. Let us now consider what happens if we start from a nonzero c. The unperturbed BRST operator Q has a connection part proportional to c, as well as deformed bilinear and trilinear brackets, as can be seen form (7.22). Now if we turn on a 2-form B, the abstract formula for the deformed BRST operator Q_B is slightly changed due to the presence of the trilinear product,

$$Q_B = Q + \{B, \cdot\} + \frac{1}{2}\{B, B, \cdot\} . \tag{7.25}$$

Moreover, we also find a correction for the bracket proportional to the trilinear bracket,

$$\{\cdot, \cdot\}_B = \{\cdot, \cdot\} + \{B, \cdot, \cdot\} . \tag{7.26}$$

We might interpret this as a covariant bracket. We can repeat much of what we know about gauge theory to this 2-form theory. There is a field strength given by

$$H = QB + \{B, B\} + \{B, B, B\}\,. \tag{7.27}$$

The equations of motion for the above Chern-Simons like theory require this field strength to vanish. Also, we have gauge invariances of the form

$$\delta_\Lambda B = Q_B \Lambda = Q\Lambda + \{B, \Lambda\} + \{B, B, \Lambda\}\,. \tag{7.28}$$

The field strength H is gauge covariant in the sense that $\delta_\Lambda H = \{H, \Lambda\}_B$. Note that the gauge transformation of H involves the covariant bracket.

8. Conclusions and outlook

We studied deformations of (topological) closed string theories from a worldsheet point of view. We saw that on-shell closed string theories have the structure of a Gerstenhaber algebra (2-algebra), which generalises off-shell to a homotopy Gerstenhaber or G_∞ algebra. Deformations of the string theory can therefore mathematically be described by a deformation of these Gerstenhaber structures. Deformations of algebras are in general encoded by the deformation complex, whose essential ingredient is the Hochschild complex. We demonstrated how the structure of the Hochschild complex can be read off from the deformations of the correlation functions of the string theory. In particular, this shows the algebraic structure of the Hochschild complex to be either a Poisson algebra or a Gerstenhaber algebra.

We found that in principle one can write down three different deformation complexes for the same vector space of operations. They correspond to the deformation of different operations in the closed string theory. In particular, the deformation of the closed string by itself deforms only the (homotopy) associative part of the (homotopy) Gerstenhaber algebra. On the other hand, closed strings that arise as the boundary theory of a topological open membrane show the deformation structure of the (homotopy) Lie algebra. A relation to $\mathrm{AdS}_3/\mathrm{CFT}_2$ may be established here, since the membrane as a deformation of the CFT on the boundary is exactly the AdS/CFT correspondence.

Whether the third possible deformation, that of the Gerstenhaber structure, has a physical interpretation remains an open question. On the one hand it appears to be related to a string theory, on the other hand ghost number conservation is probably violated. This may indicate of a breakdown of conformal invariance, which tempts to speculate about a relation with the $(1,1)$ LST.

We saw that we could define deformations for only two of the three basic operations in the closed string theory at the same time. It is not completely clear what can happen to the third operation when we deform the other two. In some cases (WDVV) it is undeformed. In other cases however the structure may get lost.

For open strings, the A_∞ structure determines a superpotential on the moduli space. The higher structure constants therefore give obstructions to the flat directions due to the higher order contributions to the superpotential. This raises the question whether in the closed string case there are similar situations, where the closed string higher structure constants give nontrivial superpotentials and therefore higher obstructions. As of yet, there are no known examples of such phenomena in the physics literature, making it unclear if we really need the full A_∞ structure in general.

Mathematically speaking, the topological open membrane describes the deformation of the algebra of multi-vector fields. A nontrivial third homotopy is found in the Lie substructure. How will this be in the generality of the operad formulation of Kontsevich [7], which can be seen as a mold for describing deformations of extended objects in string theory? In this context it is also particularly interesting to examine more closely the relation between our treatment and the geometric one of [12], in which two-dimensional open-closed field theories with very general boundary data are approached axiomatically.

The precise relation to little string theory, $(2,0)$ CFT, and M5-branes remains to be studied. The effective theory we wrote down seems more natural in 5 dimensions rather than in 6, which might indicate some relation to D4-branes. We may wonder if the relation b to 2-form gauge field and c to a 3-form background is valid on the nose. The deformation for M5-branes should be related to the total field strength $H = dB + C$. For the TOM the "field strength" h is constrained to vanish, while the field c seems to deform the algebra (or rather b and c combined). A related question is the choice of boundary conditions for the fields. In the last section we chose ψ to vanish at the boundary, leading naturally to multi-vector fields for the boundary operators. One can in fact also choose Dirichlet boundary conditions for χ instead of ψ. This leads to boundary operators naturally induced by differential forms. Whether any of these choices, or perhaps both, corresponds to a decoupling limit of M5-branes is a question for further research.

Another interesting connection can be found by relating to mathematics. The deformed L_∞ algebra of the TOM that we found, including the trilinear bracket, can be seen to be the structure of a Courant algebroid [39, 40, 41]. This is a certain fibered generalisations of a quasi-Hopf algebra (quantum group), which arose in the study of constrained quantisation. More precisely, the structure we found in the TOM is that of an exact Courant algebroid. In general, exact Courant algebroids are characterised by an element of $H^3(M,\mathbb{R})$. In our language, this corresponds to the deformation c. The construction of this class is rather analogous to the class in $H^2(M,\mathbb{R})$ of a "local line bundle" (more precisely, an algebroid of the form $TM\oplus\mathbb{R}$). When this second cohomology class is an integral class, this can be extended to a genuine global line bundle. The meaning of integrality for the third cohomology class is still mysterious, and is related to a global object for the Courant algebroid.

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Suggestions have been made that this should be a gerb. The relation of the TOM to 2-form gauge theories indeed is very suggestive in that direction. One of the authors is currently involved in further investigations along these lines [42].

The algebraic structure of the deformed TOM could also be helpful in finding a "nonabelian" generalisation of 2-form gauge theories. String theory suggests the existence of these theories in connection with multiple M5-branes. In the case of D-branes, the structure of the noncommutative gauge theory related to deformed open strings and the nonabelian gauge theory related to multiple D-branes is very similar. Analogously, we could expect the structure of multiple M5-branes and deformed M5-branes to be similar in an appropriate sense. There exist more general Courant algebroids which combine the nonabelian structure of Hopf algebras and the fibration structure of the deformed tangent space we found in the TOM. This is also very suggestive for a generalisation.

The 2-form CS theory we found as an effective theory of the TOM in the target space can be used to describe moduli spaces of flat 2-form theories. If the speculation above turns out to be correct, this can be interpreted as the moduli space of flat gerbs.

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A. Algebras up to homotopy

A *homotopy associative* or A_∞*-algebra* can be defined in terms of a derivation d acting on the tensor algebra $\mathcal{T}A = \bigoplus_{n\geq 0} A^{\otimes n}$ of a (graded) vector space A. The derivation is completely determined by the map from $\mathcal{T}A$ to A. We denote the component of d mapping the nth tensor product $A^{\otimes n}$ to A by d_n. So we have $d = d_1 + d_2 + d_3 + \cdots$. All d_k are derivations in the sense that

$$d_k(a_1, \ldots, a_{k+n}) = \sum_{i=0}^{n} (-1)^{i(k-1)} (a_1, \ldots, d_k(a_{i+1}, \ldots, a_{i+k}), \ldots, a_{k+n}) . \qquad \text{(A.1)}$$

Furthermore, d is a twisted differential in the following sense. Considering the shifted algebra $\Pi A = A[1]$.[12] The shifted maps $\tilde{d}_k = \Pi \circ d_k \circ (\Pi^{-1})^{\otimes k}$ should form a coboundary on the suspension, i.e. $\tilde{d}^2 = 0$, of degree 1. This implies an infinite number of

[12]For an integer k, $[k]$ denotes a shift of the degree of a complex $C = \bigoplus_n C^n$ by k, that is $C[k]^n := C^{k+n}$; therefore, $\Pi C^n = C^{n+1}$. Physically, the shift corresponds to descent.

homogeneous relations for the d_k: for any $n \geq 0$,

$$\sum_{k+l=n+1} (-1)^{(k-1)l} d_k \circ d_l = 0\,. \tag{A.2}$$

The map d_k has degree $2-k$. Explicitly, the first few relations read $d_1^2 = 0$, $d_1 d_2 = d_2 d_1$, $d_2^2 = -d_1 d_3 - d_3 d_1$, $d_2 d_3 - d_3 d_2 = -d_1 d_4 - d_4 d_1$. These say that d_1 is a differential on A, d_2 is a product for which d_1 is a derivation, d_3 gives a correction to the associativity of this product (d_2^2 is the associator), etc.

Homotopy Lie or L_∞*-algebras* are defined in a similar way. We also start with a (graded) space A. The only difference is that everything should be (graded) anti-symmetric; the tensor product of the algebra is replaced by the (graded) exterior product, $\bigoplus_n \bigwedge^n A$, and the products d_n are all (graded) anti-commutative. More precisely, the differential $d = \sum_n d_n$ can be considered as an operator $\tilde{d}$ operating on $\bigoplus_n S^n(A[1]) \simeq \bigoplus_n (\bigwedge^n A)[n]$ [13] by conjugating with the shift, such that $\tilde{d}^2 = 0$. They are called brackets; for example $d_2^2 = 0$ is the Jacobi identity for the Lie bracket defined by d_2.

A *Gerstenhaber algebra* (G-algebra) is a $\mathbb{Z}$-graded algebra with a graded commutative associative product $\cdot$ of degree 0 and a bracket $[\cdot,\cdot]$ of degree -1 (the Gerstenhaber bracket), which is such that $A[1]$ is a graded Lie algebra. Furthermore, the map $[\alpha,\cdot]$ must be a graded derivation of the product,

$$[\alpha, \beta\cdot\gamma] = [\alpha,\beta]\cdot\gamma + (-1)^{(|\alpha|-1)|\beta|}\beta\cdot[\alpha,\gamma]\,. \tag{A.3}$$

We can generalise this to a *differential Gerstenhaber algebra* (or *DG*) by adding a differential δ of degree 1, satisfying the graded derivation conditions with respect to the product and the bracket. Note that the shift $A[1]$ of a $(D)G$ algebra has the structure of a L_∞ algebra. Hence there is a degree one differential on $S^*(A[2])$ which squares to zero.

There does not seem to be an overall agreement over the notion of *homotopy Gerstenhaber algebra* or G_∞*-algebra* in the literature. Some possible definitions are given in [35, 16, 30]. They are fairly complicated constructions, and we will not attempt to give a definition here. We will mainly observe that they contain at least an A_∞ and a L_∞ subalgebra, with a shared differential.

B. The Hochschild cohomology of a polynomial algebra

We can give an explicit description of the Hochschild cohomology of a general polynomial algebra. Consider the algebra of polynomials in a finite number of $\mathbb{Z}$-graded variables x^i of degree $\deg(x^i) = q_i \in \mathbb{Z}$, so the space $A = \mathbb{C}[x^1,\ldots,x^N]$. We view

[13]This relation is induced by $(\Pi a_1 \otimes \cdots \otimes \Pi a_n)_S \to (-1)^{\sum_k (n-k)|a_k|} a_1 \wedge \cdots \wedge a_n$.

it as an algebra over the operad $H_*(C_d)$ (see [7]) so a d-algebra, with zero differential and zero Lie bracket. Here we assume that $d \geq 2$. The Hochschild cohomology of this algebra is, as a $\mathbb{Z}$-graded vector space, the algebra of polynomials $H^*(\mathrm{Hoch}(A)) = \mathbb{C}[x^1, \ldots, x^N, y_1, \ldots, y_N]$ in the doubled set of variables x^i, y_i, where the extra generators have degree $\deg(y_i) = d - q_i$ [7]. In general, for the algebra $\mathcal{O}(M)$ of regular functions on a smooth $\mathbb{Z}$-graded algebraic supermanifold M, the Hochschild cohomology is given by the algebra of functions on the total space of the twisted by $[d]$ cotangent bundle to M, $H^*(\mathrm{Hoch}(\mathcal{O}(M))) = \mathcal{O}(T^*[d]M)$. The proof goes along the same lines as the Hochschild-Kostant-Rosenberg theorem, which gives this result for associative algebras of functions ($d = 1$).

When the Lie bracket on the original d-algebra is nonzero, this leads to a coboundary operator on the above Hochschild cohomology. To find the actual Hochschild cohomology one should take the cohomology with respect to this coboundary. This coboundary operator is canonically related to the bracket. A bracket on the d-algebra corresponds to a Poisson structure ω^{ij} of degree $1 - d$ on M. When we use local coordinates (x^i, y_i) on $T^*[d]M$, as in the polynomial algebras above, the coboundary operator is given locally by $\omega^{ij} y_j \frac{\partial}{\partial x^i}$, which indeed has degree 1. We can also give this differential operator globally on $T^*[d]M$. We denote the pull-back of the Poisson structure ω to the full space also by ω. The total space $T^*[d]M$ has a canonical 1-form θ. This 1-form is such that the canonical symplectic structure is given by $d\theta$, and in local coordinates is given by $\theta = y_i dx^i$ (this differential form might be familiar from classical mechanics, where it is usually denoted $p_i dq^i$). Contracting the bi-vector ω with this form leads to a vector field $\theta \cdot \omega$, generating the above differential.

C. Double complexes and spectral sequences

In this appendix we shortly discuss double complexes and their cohomology. For more details see e.g. [43]. A double complex consists of a set of vector spaces $C^{p,q}$ carrying two degrees, together with two mutually anticommuting coboundary operators d and δ, so $d^2 = \delta^2 = d\delta + \delta d = 0$.[14] The operator δ increases the first degree p by one, and the d increases q by one. We can draw this double complex in a diagram as in

[14] One usually considers commuting coboundary operators, introducing extra sign factors in the formulas. It can easily seen however that this is completely equivalent.

(C.1), with the operator δ acting horizontally and d acting vertically.

$$\begin{array}{ccccccccc} C^{0,0} & \xrightarrow{\delta} & C^{1,0} & \rightarrow & C^{2,0} & \rightarrow & C^{3,0} & \rightarrow & \cdots \\ {\scriptstyle d}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ C^{0,1} & \rightarrow & C^{1,1} & \rightarrow & C^{2,1} & \rightarrow & C^{3,1} & \rightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ C^{0,2} & \rightarrow & C^{1,2} & \rightarrow & C^{2,2} & \rightarrow & C^{3,2} & \rightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \vdots & & \vdots & & \end{array} \tag{C.1}$$

To any double complex one can canonically connect a complex, where the total degree equals the sum of the two degrees, so that the degree k space of this complex is given by

$$C^k = \bigoplus_{p+q=k} C^{p,q} \,. \tag{C.2}$$

The total coboundary operator on this complex is given by $D = d + \delta$. The essential property $D^2 = 0$ can easily be checked from the analogous property of the two coboundary operators. Also, it is clear that it increases the total degree k by one. There is now a very convenient way to calculate the total cohomology $H^*_D(C)$ of this induced complex. The idea is to calculate separately the d and δ cohomology. First one calculates cohomology with respect to d,

$$E_1 = H_d(C) \,. \tag{C.3}$$

This is the first approximation to the total cohomology. The operator δ in general also induces a coboundary operation on this cohomology, which we also denote by δ. We can now make a better approximation of the total cohomology by taking the cohomology with respect to this coboundary,

$$E_2 = H_\delta(E_1) \,. \tag{C.4}$$

In general however, there may still be a coboundary operator left on the result. This procedure can be repeated, leading to a series of complexes E_r with coboundary operator d_r,

$$E_r = H_{d_r}(E_{r-1}) \,, \tag{C.5}$$

with the rth coboundary operator having degree $(r, 1-r)$. One usually find that E_r becomes stationary after a certain point. This happens for example if the range of one of the bidegrees is finite, so that d_r must vanish for sufficiently large r.

In the spectral sequence we can represent a class in the zeroth term E_0 by a d-closed element α_0. In the first term E_1 we take the cohomology with respect to δ, but in the d-cohomology. This means that α_0 should be δ-closed up to the image of d. A class in E_1 is therefore represented by a pair (α_0, α_1), such that with $d\alpha_0 = 0$, and $\delta\alpha_0 = -d\alpha_1$. Now in general, the second term E_2 has a remaining coboundary operator. The coboundary operator acting on the representing element α_0 is given by the class of $\delta\alpha_1$, $d_2[\alpha_0] = [\delta\alpha_1]$. This can be depicted as follows

$$\begin{array}{ccccc} & & & & \vdots \\ & & \alpha_1 & \to & d_2\alpha_0 \\ & & \downarrow & & \\ \alpha_0 & \to & d_1\alpha_0 & & \\ \downarrow & & & & \\ 0 & & & & \end{array} \tag{C.6}$$

where d acts vertically and δ acts horizontally. For α to represent a cohomology class in E_2, this requires $d_2\alpha$ to be zero. Remember however that we are still working in the d-cohomology, therefore it only needs to be zero as a class in this cohomology. In other words, it only needs to be zero modulo a d-exact term. This repeats the diagram above until at some point it terminates, when the differential is zero. It gives rise to a sequence of equations,

$$d\alpha_0 = 0\,, \qquad \delta\alpha_0 = -d\alpha_1\,, \qquad \delta\alpha_1 = -d\alpha_2\,, \qquad \delta\alpha_2 = -d\alpha_3\,, \qquad \ldots \tag{C.7}$$

These are the same as the familiar descent equations. It is easily checked that the total representative $\alpha = \alpha_0 + \alpha_1 + \cdots$ is closed with respect to total coboundary D.

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